數值偏微分法

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- 回顧
  - 微分方程和商品評價的關係
    - Black Scholes PDE
- 偏微分方程求解
  - Explicit method:和樹狀結構相似
    - Stable conditions
    - 變數變換
  - Implicit method
  - Crank-Nicolson Method:前兩者的結合

使用Ito's formula 推導衍生性商品價格P的隨機過程

- By Black-Scholes assumption $dS=(\mu-q)Sdt+\sigma Sdz$ where dz is a standard Brownian motion.
- By Ito lemma

$$dP=\frac{\partial P}{\partial t}dt+\frac{\partial P}{\partial S}dS+\frac{1}{2}\frac{\partial^2 P}{\partial S^2}(dS)^2$$

$$=[(\mu-q)S\frac{\partial P}{\partial S}+\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}]dt+\sigma S\frac{\partial P}{\partial S}dz$$

Black-Scholes Differential Equation

$$\begin{cases} dS=(\mu-q)Sdt+\sigma Sdz \\ dP=\left((\mu-q)S\frac{\partial P}{\partial S}+\frac{\partial P}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt+\sigma S\frac{\partial P}{\partial S}dz \end{cases}$$

$$\rightarrow dP-\frac{\partial P}{\partial S}dS=\left(\frac{\partial P}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt$$

short: 1 derivative security
long: $\frac{\partial P}{\partial S}$ share underlying asset
the value of this portfolio $\Pi=-P+S\frac{\partial P}{\partial S}$

$$d\Pi=-dP+\frac{\partial P}{\partial S}dS=\left(-\frac{\partial P}{\partial t}-\frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt$$

$$d\Pi+qS\frac{\partial P}{\partial S}dt\rightarrow\left(-\frac{\partial P}{\partial t}-\frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt+qS\frac{\partial P}{\partial S}dt=r\Pi dt$$

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+(r-q)S\frac{\partial P}{\partial S}-rP+\frac{\partial P}{\partial t}=0$$

選擇權價格滿足此偏微分方程

美式賣權的PDE及邊界條件

The Black-Scholes differential equation for American puts is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP + \frac{\partial P}{\partial t} = 0$$

where $P(S,T) = \max(X-S, 0)$ and

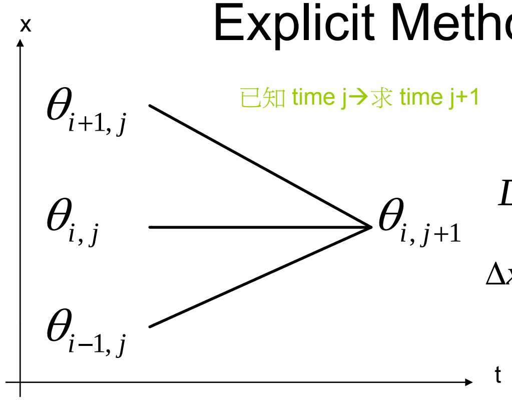
$P(S,t) = \max(\overline{P}(S,t), X-S)$ for $t < T$.

$\overline{P}$ denotes the option value at time t if it is not exercised for the next instant of time.

使用數值方法解偏微分方程

- 並非所有的偏微分方程都有公式解
- 對於無法求公式解的方程,可用數值方法逼近
  - Explicit method: 和樹狀結構評價法接近
  - Implicit method: 解方程式
  - Crank-Nicolson Method: 上述兩種方法的混合

Explicit Methods簡介



已知 time j→求 time j+1

考慮

$$D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$$

$\Delta x \equiv x_i - x_{i-1}$ $\Delta t \equiv t_{j+1} - t_j$

$$\frac{\partial\theta(x,t)}{\partial t}\Big|_{t=t_j} = \frac{\theta(x_i,t_{j+1}) - \theta(x_i,t_j)}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2\theta(x,t)}{\partial x^2}\Big|_{x=x_i} = \frac{\theta(x_{i+1},t_j) - 2\theta(x_i,t_j) + \theta(x_{i-1},t_j)}{(\Delta x)^2} + O[(\Delta x)^2]$$

代入 $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$

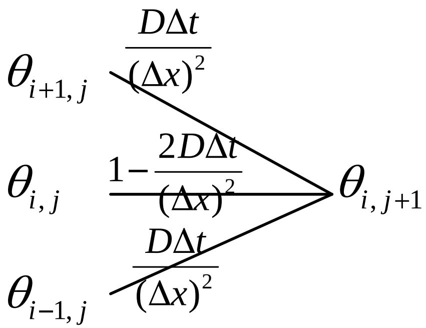
$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}$$

Rearrange as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\theta_{i+1,j} + [1 - \frac{2D\Delta t}{(\Delta x)^2}]\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\theta_{i-1,j}$$

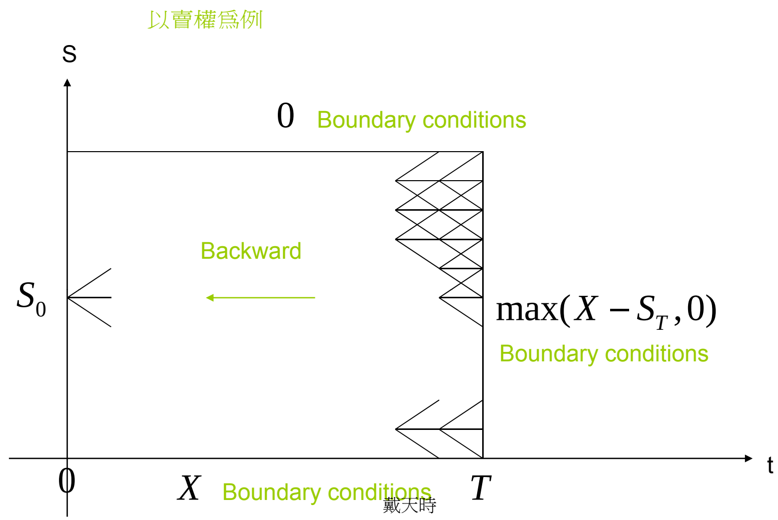
The explicit method is numerically unstable unless

$$\Delta t \le (\Delta x)^2/(2D)$$



When the stability condition is satisfied, the coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$,and $\theta_{i-1,j}$ all lie between zero and one and sum to one. They can therefore be interpreted as probabilities.

使用Explicit Method 評價選擇權



改寫Black Scholes PDE

The Black-Scholes differential equation is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP + \frac{\partial P}{\partial t} = 0$$

$$\frac{1}{2}\sigma^2 S^2 \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{(\Delta S)^2} + (r-q)S\frac{P_{i+1,j} - P_{i-1,j}}{2\Delta S} - rP_{i,j} + \frac{P_{i,j} - P_{i,j-1}}{\Delta t} = 0$$

for $1 \le i \le N-1$ $\quad P_{i,j-1} = \max(aP_{i-1,j} + bP_{i,j} + cP_{i+1,j}, X - S_{i,j-1})$

where　　已知 time j→求 time j-1

$$a \equiv [(\frac{\sigma S}{\Delta S})^2 - \frac{(r-q)S}{\Delta S}]\frac{\Delta t}{2} \qquad b \equiv 1 - r\Delta t - (\frac{\sigma S}{\Delta S})^2 \Delta t \qquad c \equiv [(\frac{\sigma S}{\Delta S})^2 + \frac{(r-q)S}{\Delta S}]\frac{\Delta t}{2}$$

Stable Conditions

Stable→ $a > 0$ $\quad b > 0$ $\quad c > 0$

$$\frac{S}{\Delta S} > \frac{r-q}{\sigma^2} \qquad \frac{1}{r + (\frac{\sigma S}{\Delta S})^2} > \Delta t$$

Since a>0　　Since b>0

如果boundary condition 沒取好 → S的值太大或太小導致a 或b 小於零 -→ 出現不合理的結果

	A	B	C	D	E	F	G	H	I	J	K	L
1	S	X	r	vol	T	delta S	delta t	q				
2	50	50	0.1	0.4	0.41667	5	0.04167	0				
3	*Stock price*					*Time to maturity (months)*						
4	*(dollars)*	*5*	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
5	*100*	0	0	0	0	0	0	0	0	0	0	0
6	*95*	0.06	0	0	0	0	0	0	0	0	0	0
7	*90*	-0.11	0.05	0	0	0	0	0	0	0	0	0
8	*85*	0.28	-0.046542	0.05	0	0	0	0	0	0	0	0
9	*80*	-0.13	0.2	0	0.05	0	0	0	0	0	0	0
10	*75*	0.46	0.06	0.2	0.041625	0.06	0	0	0	0	0	0
11	*70*	0.32	0.46	0.23	0.25	0.1	0.09	0	0	0	0	0
12	*65*	0.91	0.68	0.63	0.44	0.37	0.21	0.14	0	0	0	0
13	*60*	1.48	1.37	1.17	1.021225	0.81	0.65	0.42	0.27	0	0	0
14	*55*	2.59	2.39	2.21	1.99	1.77	1.5	1.24	0.9	0.59	0	0
15	*50*	4.26	4.08	3.89	3.68	3.44	3.18	2.87	2.53	2.07	1.5625	0
16	*45*	6.76	6.61	6.47	6.31	6.15	5.96	5.75	5.5	5.24	5	5
17	*40*	10.28	10.2	10.13	10.06	10.01	10	10	10	10	10	10
18	*35*	15	15	15	15	15	15	15	15	15	15	15
19	*30*	20	20	20	20	20	20	20	20	20	20	20
20	*25*	25	25	25	25	25	25	25	25	25	25	25
21	*20*	30	30	30	30	30	30	30	30	30	30	30
22	*15*	35	35	35	35	35	35	35	35	35	35	35
23	*10*	40	40	40	40	40	40	40	40	40	40	40
24	*5*	45	45	45	45	45	45	45	45	45	45	45
25	*0*	50	50	50	50	50	50	50	50	50	50	50

S太大導致b值爲負

	a	b	c
C8	0.927917	0.930833	0.99875
K15	0.455	0.035833	0.505
E10	0.71875	-0.504167	0.78125

如何修改?

$$\frac{1}{r+(\frac{\sigma S}{\Delta S})^2} > \Delta t = 0.04167$$ 當S太大,左式不能滿足

- 將切的期數增加(Δt↓)
- 參見 hull18.2-2.xls

$$\frac{1}{r+(\frac{\sigma S}{\Delta S})^2} \xrightarrow{\text{最大值 (S=95)}} \frac{1}{0.1+\left(\frac{0.4\times 95}{5}\right)^2} = 0.017283 > \Delta t = 0.0104$$

PDE的變數變換

Change of variable $V \equiv \ln S$ $\quad U(V,t) \equiv P(e^V, t)$

$$\frac{\partial P}{\partial t} = \frac{\partial U}{\partial t} \qquad \frac{\partial P}{\partial S} = \frac{1}{S}\frac{\partial U}{\partial V} \qquad \frac{\partial^2 P}{\partial^2 S} = \frac{1}{S^2}\frac{\partial^2 U}{\partial V^2} - \frac{1}{S^2}\frac{\partial U}{\partial V}$$

$$\frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial V^2} + (r - q - \frac{\sigma^2}{2})\frac{\partial U}{\partial V} - rU + \frac{\partial U}{\partial t} = 0$$

subject to $U(V,T) = \max(X - e^V, 0)$ and

$$U(V,t) = \max(\overline{U}(V,t), X - e^V) \qquad t < T$$

繼續持有的價值 提前履約的價值

程式撰寫

Along the V axis, the grid will span from $V_{\min}$ to $V_{\min} + N \times \Delta V$ at ΔV apart for some suitably small $V_{\min}$; hence boundary conditions at the lower $(V = V_{\min})$ and upper $(V = V_{\min} + N \times \Delta V)$ boundaries will have to be specified.

請參見 hull18.3.xls

The Explicit Scheme

The Black-Scholes differential equation is

$$\frac{1}{2}\sigma^2\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{(\Delta V)^2}+(r-q-\frac{\sigma^2}{2})\frac{U_{i+1,j}-U_{i-1,j}}{2\Delta V}-rU_{i,j}+\frac{U_{i,j}-U_{i,j-1}}{\Delta t}=0$$

for $1 \le i \le N-1$

$$U_{i,j-1}=aU_{i-1,j}+bU_{i,j}+cU_{i+1,j}$$

where

$$a \equiv [(\frac{\sigma}{\Delta V})^2-\frac{r-q-\sigma^2/2}{\Delta V}]\frac{\Delta t}{2} \qquad b \equiv 1-r\Delta t-(\frac{\sigma}{\Delta V})^2\Delta t \qquad c \equiv [(\frac{\sigma}{\Delta V})^2+\frac{r-q-\sigma^2/2}{\Delta V}]\frac{\Delta t}{2}$$

Stable→ $a>0$ $b>0$ $c>0$

$$\frac{\sigma^2}{r-q-\frac{\sigma^2}{2}}>\Delta V \qquad \frac{1}{r+(\frac{\sigma}{\Delta V})^2}>\Delta t$$

Since a>0 Since b>0

優點: a,b,c>0的條件和S (or V) 無關，取boundary condition不用考慮a or b <0 的問題
只要△V和△t夠小即可

Deep in the Money 邊界條件

. 由於取指數原因，S 的lower bound >0

$U_{0,j}$不能直接設成X

Deep in the money

Put : $\frac{\partial P}{\partial S}$ 趨近於-1 $\frac{U_{1,j}-U_{0,j}}{\Delta S}=-1$

for j : $\Delta S=U_{0,j}-U_{1,j}=e^{V_{\min}+\Delta V}-e^{V_{\min}}$

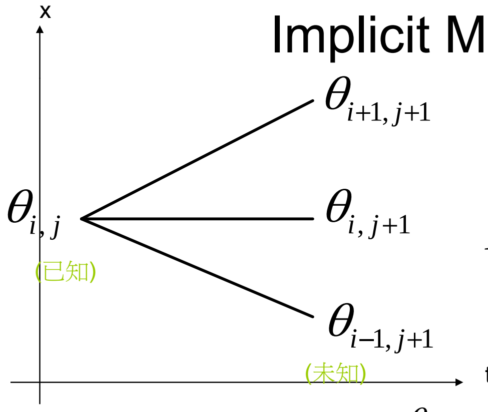
$$U_{0,j}=U_{1,j}+(e^{V_{\min}+\Delta V}-e^{V_{\min}})$$

Deep OTM 邊界條件

For the upper boundary, we set $U_{N,j-1}=0$.

$U_{i,j}$ is set to the greater of the value derived above and $X-e^{V_{\min}+i\times\Delta V}$ for early-exercise considerations.

Implicit Methods



考慮

$$D(\partial^2\theta/\partial x^2)-(\partial\theta/\partial t)=0$$

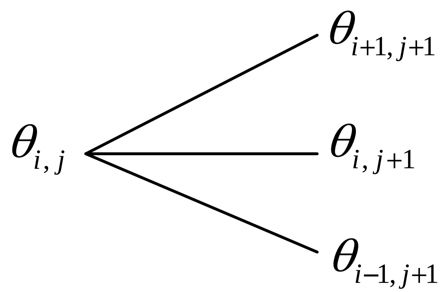
$$\Delta x\equiv x_i-x_{i-1} \qquad \Delta t\equiv t_{j+1}-t_j$$

$$\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta t}=D\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{(\Delta x)^2}$$

Rearrange as

$$\theta_{i-1,j+1}-(2+\gamma)\theta_{i,j+1}+\theta_{i+1,j+1}=-\gamma\theta_{i,j}$$

where $\gamma\equiv(\Delta x)^2/(D\Delta t)$



The value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known.

使用解方程式的方法求值

Suppose the boundary conditions are given at $x=x_0$ and $x=x_{N+1}$.

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & a & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & a \end{bmatrix}\begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix}=\begin{bmatrix} -\gamma\theta_{1,j}-\theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j}-\theta_{N+1,j+1} \end{bmatrix}$$

where $a\equiv-2-\gamma$

Crank-Nicolson Method

Taking the average of explicit method and implicit method results in

$$\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta t}=\frac{1}{2}[D\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{(\Delta x)^2}+D\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{(\Delta x)^2}]$$

Rearrange as

$$\gamma\theta_{i,j+1}-\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{2}=\gamma\theta_{i,j}+\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{2}$$

使用解方程式的方法求值

Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.

$$\begin{bmatrix} a & -0.5 & 0 & \cdots & \cdots & \cdots & 0 \\ -0.5 & a & -0.5 & 0 & \cdots & \cdots & 0 \\ 0 & -0.5 & a & -0.5 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -0.5 & a & -0.5 \\ 0 & \cdots & \cdots & \cdots & 0 & 0.5 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} \frac{\theta_{0,j} + (2\gamma-2)\theta_{1,j} + \theta_{2,j} + \theta_{0,j+1}}{2} \\ \frac{\theta_{1,j} + (2\gamma-2)\theta_{2,j} + \theta_{3,j}}{2} \\ \frac{\theta_{2,j} + (2\gamma-2)\theta_{3,j} + \theta_{4,j}}{2} \\ \vdots \\ \vdots \\ \frac{\theta_{N-2,j} + (2\gamma-2)\theta_{N-1,j} + \theta_{N,j}}{2} \\ \frac{\theta_{N-1,j} + (2\gamma-2)\theta_{N,j} + \theta_{N+1,j} + \theta_{N+1,j+1}}{2} \end{bmatrix}$$

where $a \equiv 1 + \gamma$

使用Implicit Method解變數變換後的 Black Scholes PDE

$$\frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial V^2} + (r - q - \frac{\sigma^2}{2})\frac{\partial U}{\partial V} - rU + \frac{\partial U}{\partial t} = 0$$

改寫

$$\frac{1}{2}\sigma^2 \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta V)^2} + (r - q - \frac{\sigma^2}{2})\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta V} - rU_{i,j} + \frac{U_{i,j+1} - U_{i,j}}{\Delta t} = 0$$

for $1 \le i \le N-1$

$$aU_{i-1,j} + bU_{i,j} + cU_{i+1,j} = U_{i,j+1}$$

where

$$a \equiv [-(\frac{\sigma}{\Delta V})^2 + \frac{r - q - \sigma^2/2}{\Delta V}]\frac{\Delta t}{2} \qquad b \equiv 1 + r\Delta t + (\frac{\sigma}{\Delta V})^2 \Delta t$$

$$c \equiv -[(\frac{\sigma}{\Delta V})^2 + \frac{r - q - \sigma^2/2}{\Delta V}]\frac{\Delta t}{2}$$

The system of equations can be written in matrix form:

$$\begin{bmatrix} b^* & c & 0 & \cdots & \cdots & \cdots & 0 \\ a & b & c & 0 & \cdots & \cdots & 0 \\ 0 & a & b & c & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & a & b & c \\ 0 & \cdots & \cdots & \cdots & 0 & a & b \end{bmatrix} \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ U_{3,j} \\ \vdots \\ \vdots \\ \vdots \\ U_{N-1,j} \end{bmatrix} = \begin{bmatrix} U_{1,j+1} - K \\ U_{2,j+1} \\ U_{3,j+1} \\ \vdots \\ \vdots \\ U_{N-2,j+1} \\ U_{N-1,j+1} \end{bmatrix}$$

where $b^* \equiv a + b$ and $K \equiv a(e^{V_{\min}+\Delta V} - e^{V_{\min}})$.

(See the proof in the next slide)

$$aU_{0,j} + bU_{1,j} + cU_{2,j} = U_{1,j+1}$$

$$(a+b)U_{1,j} + cU_{2,j} = U_{1,j+1} + a(U_{1,j} - U_{0,j})$$

Deep in the money $\frac{\partial P}{\partial S} \to -1$ $\qquad \frac{U_{1,j} - U_{0,j}}{\Delta S} = -1$

$$(a+b)U_{1,j} + cU_{2,j} = U_{1,j+1} - a(e^{V_{\min}+\Delta V} - e^{V_{\min}})$$

Define $b^* \equiv a + b \qquad K \equiv a(e^{V_{\min}+\Delta V} - e^{V_{\min}})$

$$b^* U_{1,j} + cU_{2,j} = U_{1,j+1} - K$$