Using The LIBOR Market Model to Price The Interest Rate Derivatives: A Recombining Binomial Tree Methodology

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AGENDA

- Introduction
- Review of interest rate models
- Market conventions of LMM and the discrete-time version of LMM
- Introducing the HSS recombining node methodology and applying to the LIBOR market model
- The pricing of the interest rate derivatives in LMM
- Conclusions

INTRODUCTION

- We can't observe the instantaneous short rate and instantaneous forward rate in the traditional interest rate models
- We adapt LIBOR market model which is based on the forward LIBOR rate that we can observe from the daily market
- When implementing LMM in lattice method, we face the explosive tree due to the non-Markov property

INTRODUCTION

- The nonrecombining node make our pricing procedure inefficient and don’t satisfy the market requirement
- We apply the HSS methodology into LMM that make the nodes combine and make the pricing the derivatives feasible
- The method we proposed make our valuation more efficient and more accurate

REVIEW OF INTEREST RATE MODELS

- Equilibrium models
- No-arbitrage models
  - Instantaneous short rate models
  - Instantaneous forward rate model
  - Forward rate model

FORWARD RATE MODEL
LIBOR MARKET MODEL (LMM)

- The LMM was discovered by Brace, Gatarek, and Musiela (1997) and was initially referred to as the BGM model by practitioners
- There are two commonly used versions of the LMM:
  - one is the lognormal forward LIBOR model (LFM) for pricing caps
  - the other is the lognormal swap model (LSM) for pricing swaptions

FORWARD RATE MODEL
LIBOR MARKET MODEL (LMM)

- The LFM specifies the forward rate $f(t;T_i,T_{i+1})$ following zero-drift stochastic process under its own forward measure:

$$\frac{df(t;T_i,T_{i+1})}{f(t;T_i,T_{i+1})} = \sigma_i(t)dW_i(t)$$

  - $dW_i(t)$ is a Brownian motion under the forward measure $\tilde{P}^i$ defined with respect to the numeraire asset
  - $\sigma_i(t)$ measures the volatility of the forward rate process

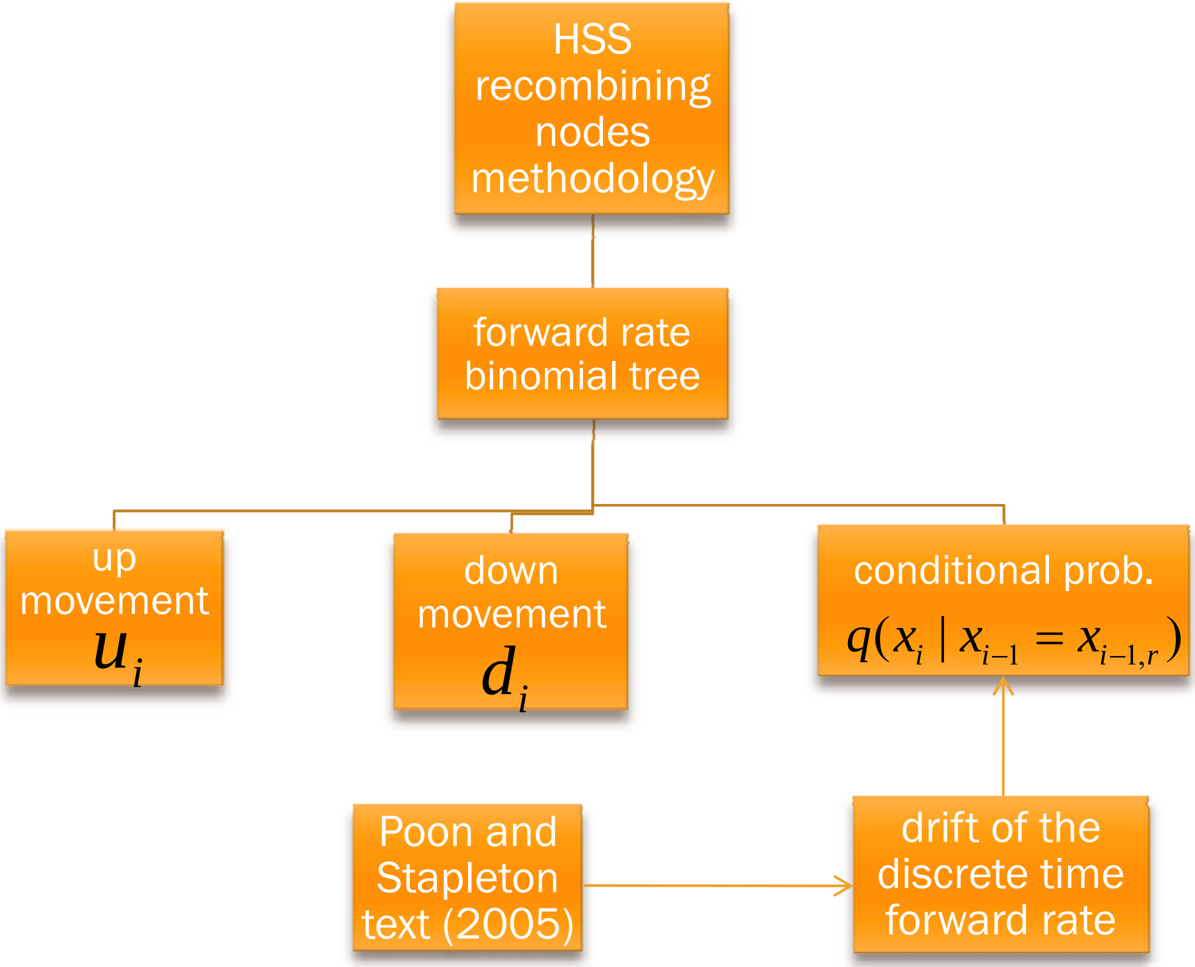
MARKET CONVENTIONS OF THE LMM AND THE DISCRETE-TIME VERSION OF THE LMM

- The relationship between the discrete LIBOR rate $L(T_i, T_{i+1})$ for the term $\delta_i = T_{i+1} - T_i$ and the zero coupon bond price $P(T_i, T_{i+1})$, given as follows:

$$P(T_i, T_{i+1}) = \frac{1}{1 + \delta_i L(T_i, T_{i+1})}$$

where $t = T_0 < T_1 < T_2 < \cdots < T_n$ is the time line and δ_i is called the tenor for the period T_i to T_{i+1}

METHODOLOGY



INTRODUCING THE HSS RECOMBINING NODE METHODOLOGY

- Ho, Stapleton and Subrahmanyam (1995) [HSS] suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process
- Our assumption about LMM satisfies the conditions in the HSS methodology and apply it into LMM

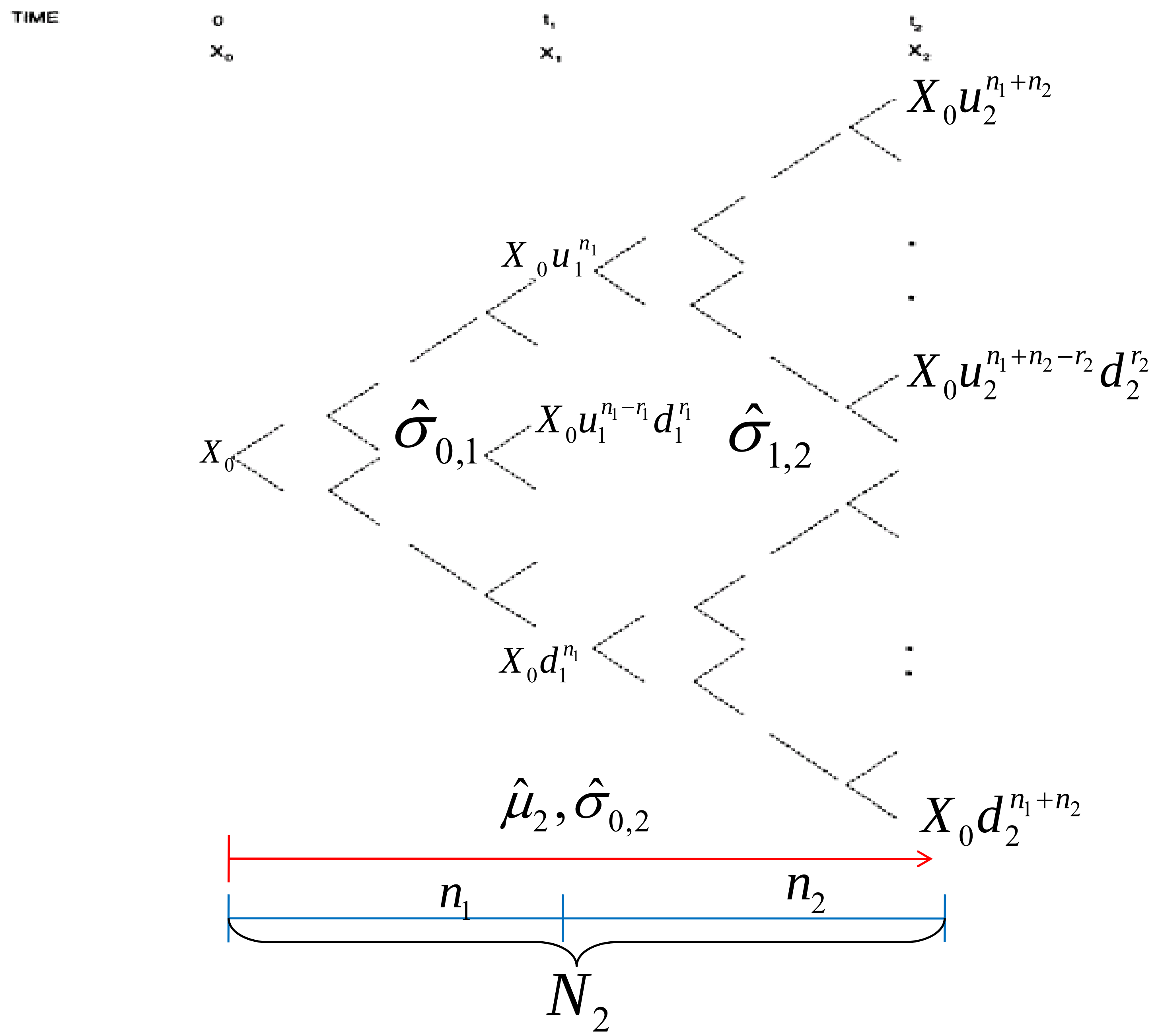
HSS METHODOLOGY

- HSS assume the price of underlying asset follows a lognormal diffusion process:

$$d \ln X(t) = \mu(X(t), t)dt + \sigma(t)dW(t)$$

where μ and σ are the instantaneous drift and volatility of $\ln X$, and $dW(t)$ is a standard Brownian motion

  - They denote the unconditional mean of the logarithmic asset return at time i as μ_i
  - The conditional volatility over the period i -1 to i is denoted as $\sigma_{i-1,i}$
  - The unconditional volatility is $\sigma_{0,i}$


TIME
0
X_0
t_1
X_1
t_2
X_2
$X_0u_2^{n_1+n_2}$
$X_0u_1^{n_1}$
$X_0u_2^{n_1+n_2-r_2}d_2^{r_2}$
X_0
$\hat{\sigma}_{0,1}$
$X_0u_1^{n_1-r_1}d_1^{r_1}$
$\hat{\sigma}_{1,2}$
$X_0d_1^{n_1}$
$\hat{\mu}_2, \hat{\sigma}_{0,2}$
$X_0d_2^{n_1+n_2}$
n_1
n_2
N_2

HSS METHODOLOGY

- HSS method involves the construction of m separate binomial distribution, where the time periods are denoted $t_1, \cdots, t_i, \cdots, t_m$
- And have the set of a discrete stochastic for X_i, where X_i is only defined at time t_i
- The general form of X_i at node r

$$X_{i,r} = X_0 u_i^{N_i - r} d_i^r$$

where $N_i = \sum_{l=1}^{i} n_l$

HSS METHODOLOGY

- They denote $x_i = \ln(X_i / X_0)$ and the probabilities to reach x_i given a node $x_{i-1,r}$ at t_{i-1} as

$$q(x_i \mid x_{i-1} = x_{i-1,r}) \ or \ q(x_i)$$

- **Lemma 1** Suppose that the up and down movements u_i and d_i are chosen so that

$$d_i = \frac{2(E(X_i)/X_0)^{\frac{1}{N_i}}}{1+\exp(2\sigma_{i-1,i}\sqrt{(t_i - t_{i-1})/n_i})}, \quad i = 1, 2, \cdots, m,$$

$$u_i = 2(E(X_i)/X_0)^{\frac{1}{N_i}} - d_i, \quad i = 1, 2, \cdots, m,$$

HSS METHODOLOGY

Theorem

Suppose that the X_i are joint lognormally distributed. If the X_i are approximated with binomial distributions with $N_i = N_{i-1} + n_i$ stages and u_i and d_i given by the above, and if the conditional probability of an up movement at node r at time $i-1$ is

$$q(x_i \mid x_{i-1} = x_{i-1,r}) = \frac{a_i + b_i x_{i-1,r} - (N_{i-1} - r)\ln u_i - r\ln d_i}{n_i(\ln u_i - \ln d_i)} - \frac{\ln d_i}{\ln u_i - \ln d_i}, \quad \forall i,\ r$$

then $\hat{\mu}_i \rightarrow \mu_i$ and $\hat{\sigma}_{0,i} \rightarrow \sigma_{0,i}$ and $\hat{\sigma}_{i-1,i} \rightarrow \sigma_{i-1,i}$ as $n_i \rightarrow \infty, \forall i$

APPLYING HSS METHODOLOGY INTO LMM

- We apply this methodology into the LMM and make some change to satisfy our conventions
- We have the following propositions

PROPOSITION 1

- For the forward LIBOR rate which follows the lognormal distribution, we can choose the proper up and down movements to determine the i-th period of the T_n-maturity forward LIBOR rate and have the form

$$f(i;T_n,T_{n+1})_r = f(0;T_n,T_{n+1})u_i^{N_i-r}d_i^r,\ i=T_1,T_2,\cdots,T_n$$

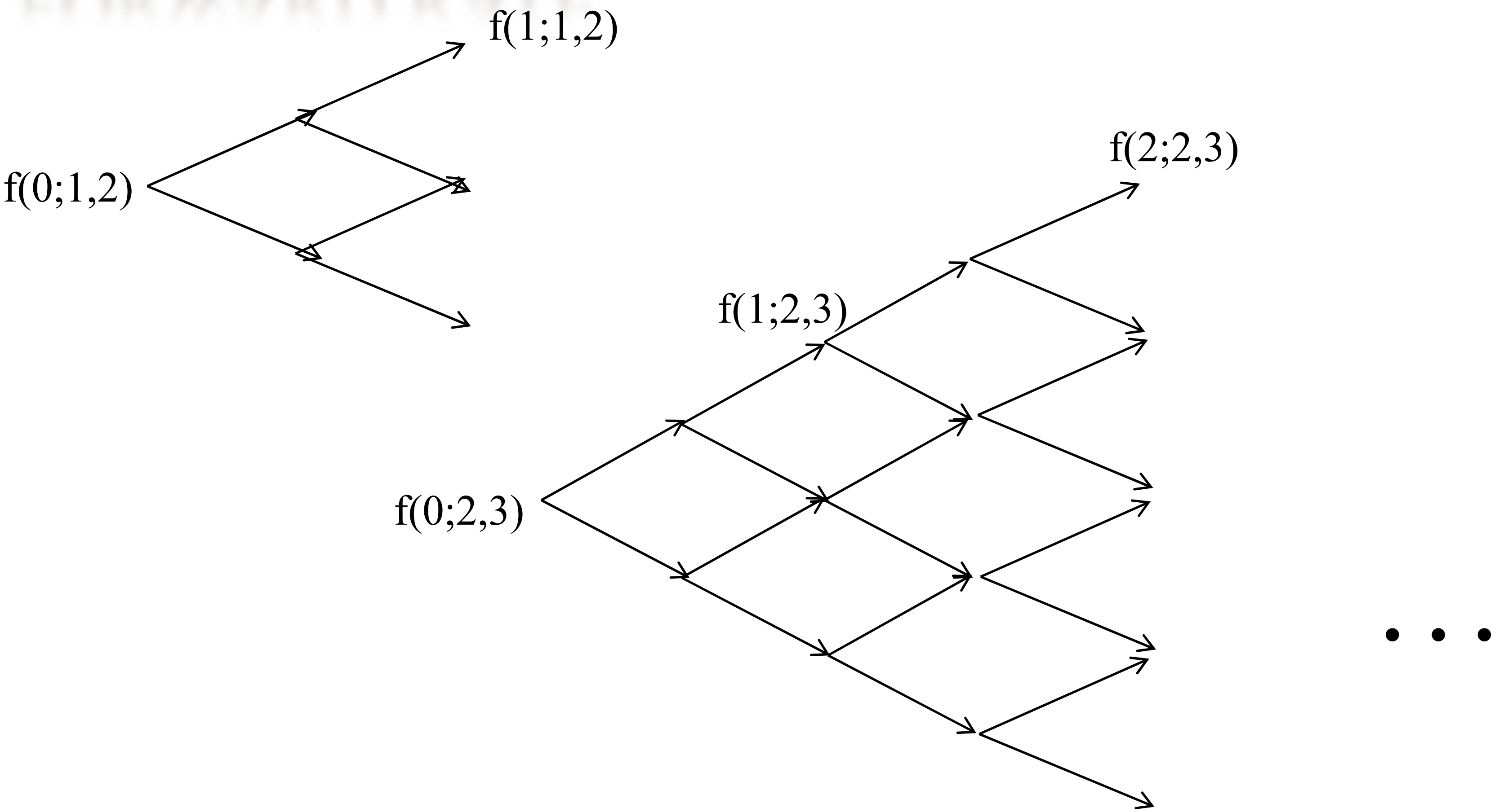
where
$$d_i = \frac{2[E(f(i;T_n,T_{n+1}))/f(0;T_n,T_{n+1})]^{\frac{1}{N_i}}}{1+\exp(2\sigma_{i-1,i}\sqrt{(T_i-T_{i-1})/n_i})}$$

$$u_i = 2[E(f(i;T_n,T_{n+1}))/f(0;T_n,T_{n+1})]-d_i$$

$$N_i = N_{i-1}+n_i$$

r: node's number from top to bottom at time T_i

THE BINOMIAL TREES IN DISCRETE TIME FORWARD RATE



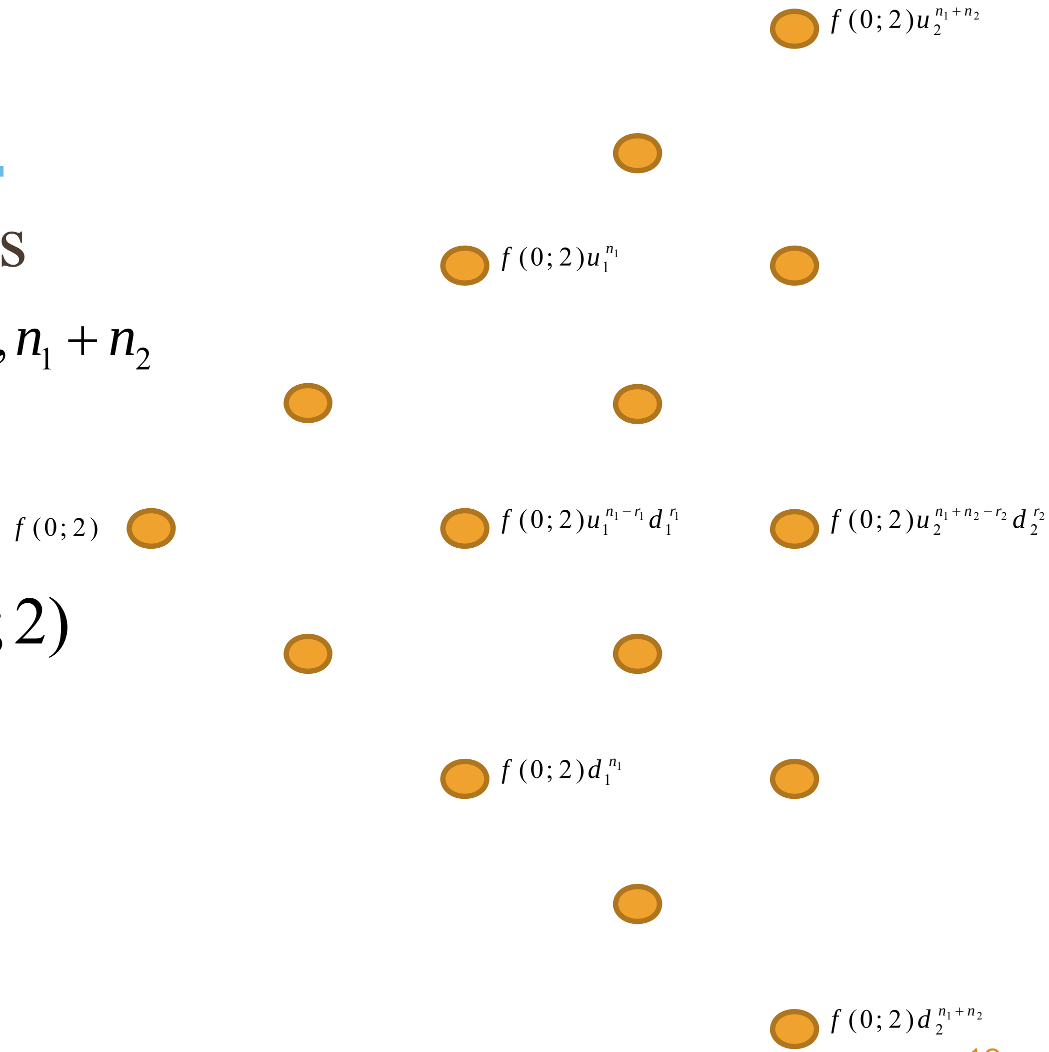
THE STRUCTURE OF THE BINOMIAL TREE

- With $n_1 + 1$ nodes at T_1 numbered $r = 0, 1, \ldots, n_1$.
  There are $n_1 + n_2 + 1$ nodes at T_2 numbered $r = 0, 1, \ldots, n_1 + n_2$
  Here we write the forward rate $f(0; 2, 3)$

  in abbreviated form $f(0; 2)$

  and take $n_1 = n_2 = 2$,

  $0 \le r_1 \le n_1,\ 0 \le r_2 \le n_1 + n_2$



PROPOSITION 2

- Suppose that the forward LIBOR rate $f(i;T_n,T_{n+1})$ are joint lognormally distributed. If the $f(i;T_n,T_{n+1}),\ i=T_1,T_2,\cdots,T_n$ are approximated with binomial distributions with $N_i = N_{i-1} + n_i$ stages and u_i and d_i given by proposition 1, and if the conditional probability of an up movement at node *r* at time T_i is

$$q(x_i \mid x_{i-1} = x_{i-1,r}) = \frac{E_{i-1}(x_i) - (N_{i-1} - r)\ln u_i - r\ln d_i}{n_i(\ln u_i - \ln d_i)} - \frac{\ln d_i}{\ln u_i - \ln d_i} \quad \forall\ i,\ r$$

PROPOSITION 2

where

$$x_i = \ln \frac{f(i;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}$$

$$E_{i-1}(x_i) = E(x_i) - b_i E(x_{i-1}) + b_i x_{i-1,r}$$

$$b_i = \sqrt{[t_i\sigma_{0,i}^2 - (t_i - t_{i-1})\sigma_{i-1,i}^2]/t_{i-1}\sigma_{0,i-1}^2}$$

- To determining the conditional probability, it has some skills to use for the term $E_{i-1}(x_i)$
- We first derive term $E(x_i)$. Since forward rate is lognormally distributed, we have

$$E(x_i) = \ln[\frac{E(f(i;T_n,T_{n+1}))}{f(0;T_n,T_{n+1})}] - \frac{1}{2}\sigma_{0,i}^2$$

THE DERIVATION OF EXPECTATION

- The most important result in Poon and Stapleton text is the drift of the discrete time forward rate and we rewrite as

$$\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})} = 1 + \frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)} \cdot \tilde{\sigma}_{1,n} + \frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)} \cdot \tilde{\sigma}_{2,n} + \cdots + \frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})} \cdot \tilde{\sigma}_{n,n}$$

- Then multiple the $\dfrac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}$ term on both side to get the general form of $E_t(f(T_1;T_n,T_{n+1}))/f(0;T_n,T_{n+1})$

THE DERIVATION OF EXPECTATION

- The general form of expectation is given as

$$\frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})}=$$

$$\frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times(1+\frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n}+\frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n}+\cdots+\frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n})$$

SPECIAL CASE

- When n_l stages approach infinite $l = 1, \cdots, i$, the sum of n_l stages also approach infinite, that is
  $N_i = \sum_{l=1}^{i} n_l \to \infty$
- We can reduce the up and down movements to the briefer form which is easier to calculate and show as follows

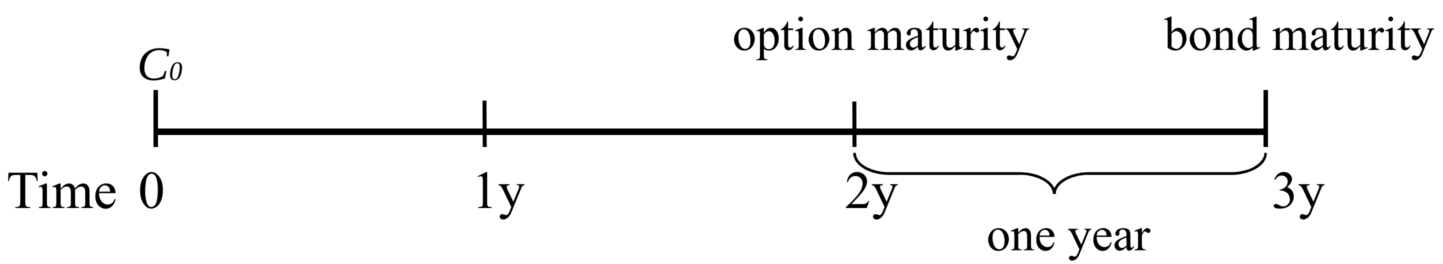
$$d_i = \frac{2}{1 + \exp(2\sigma_{i-1,i}\sqrt{(T_i - T_{i-1})/n_i})}$$

$$u_i = 2 - d_i$$

- The conditional probability $q(x_l) \to 0.5$ as $n_l \to \infty$

THE VALUATION OF EMBEDDED OPTION ON ZERO COUPON BOND IN LMM

- The embedded option on ZCB is a bond that can be callable before maturity date with a callable price K



- We have a three years maturity zero coupon bond with a callable price K equal to 0.952381 dollar at year two

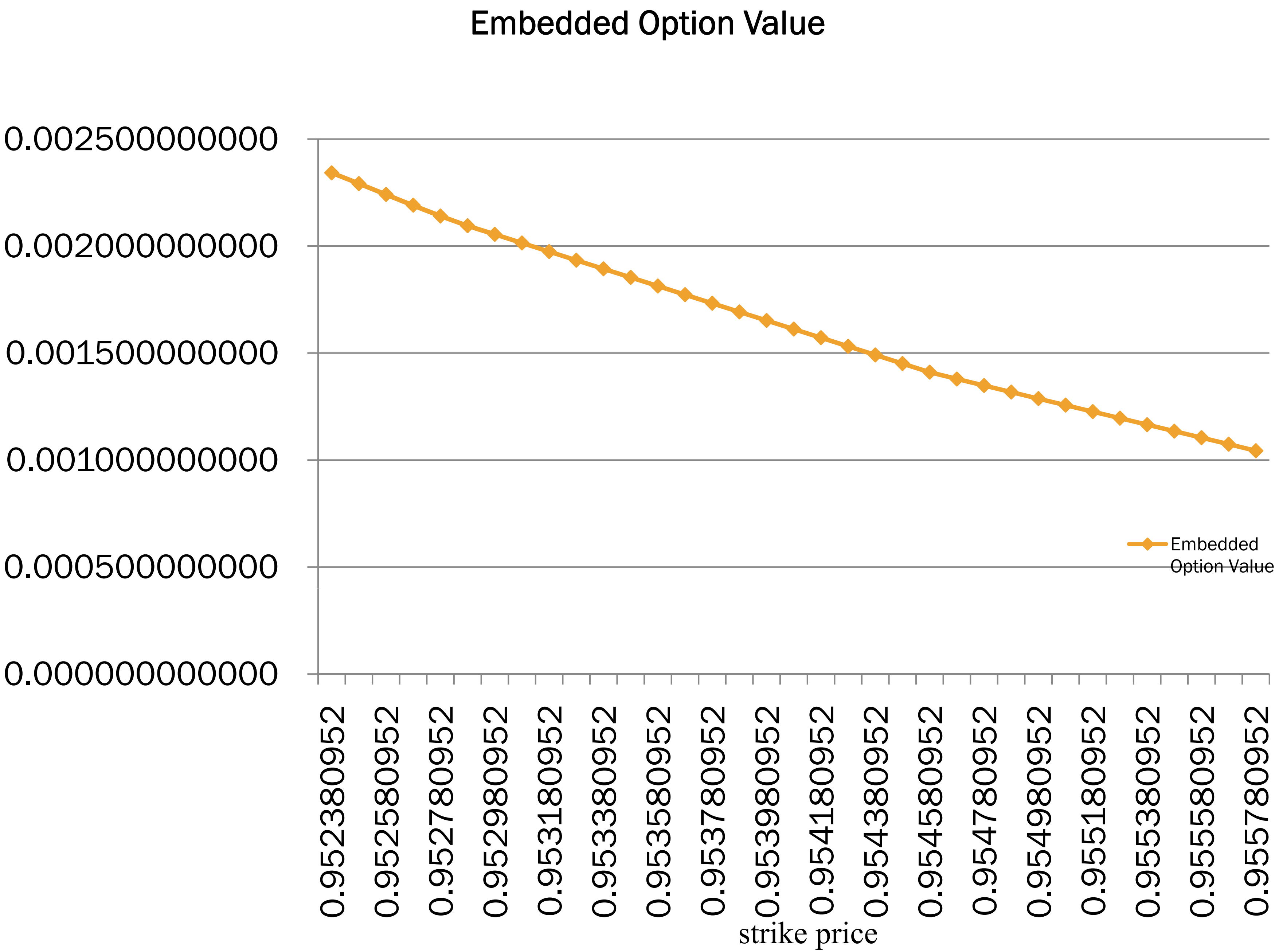
THE VALUATION OF EMBEDDED OPTION ON ZERO COUPON BOND IN LMM

- Pricing the option value C_0 of this callable bond at time 0
- Here, we take the flat forward rate 5% and constant volatility 10%

$$
\begin{aligned}
C_0 &= P(0,2) \times E[\max(P(2,3) - K, 0)] \\
&= 0.90702948 \times 0.00258128 \\
&= 0.00234130
\end{aligned}
$$

SENSITIVE ANALYSIS – STRIKE PRICE

- We change strike price from 0.952381 to 0.96 which vary 0.001 to see the relationship between strike price and embedded option value
- We find that when the strike price increases, the embedded option value decreases

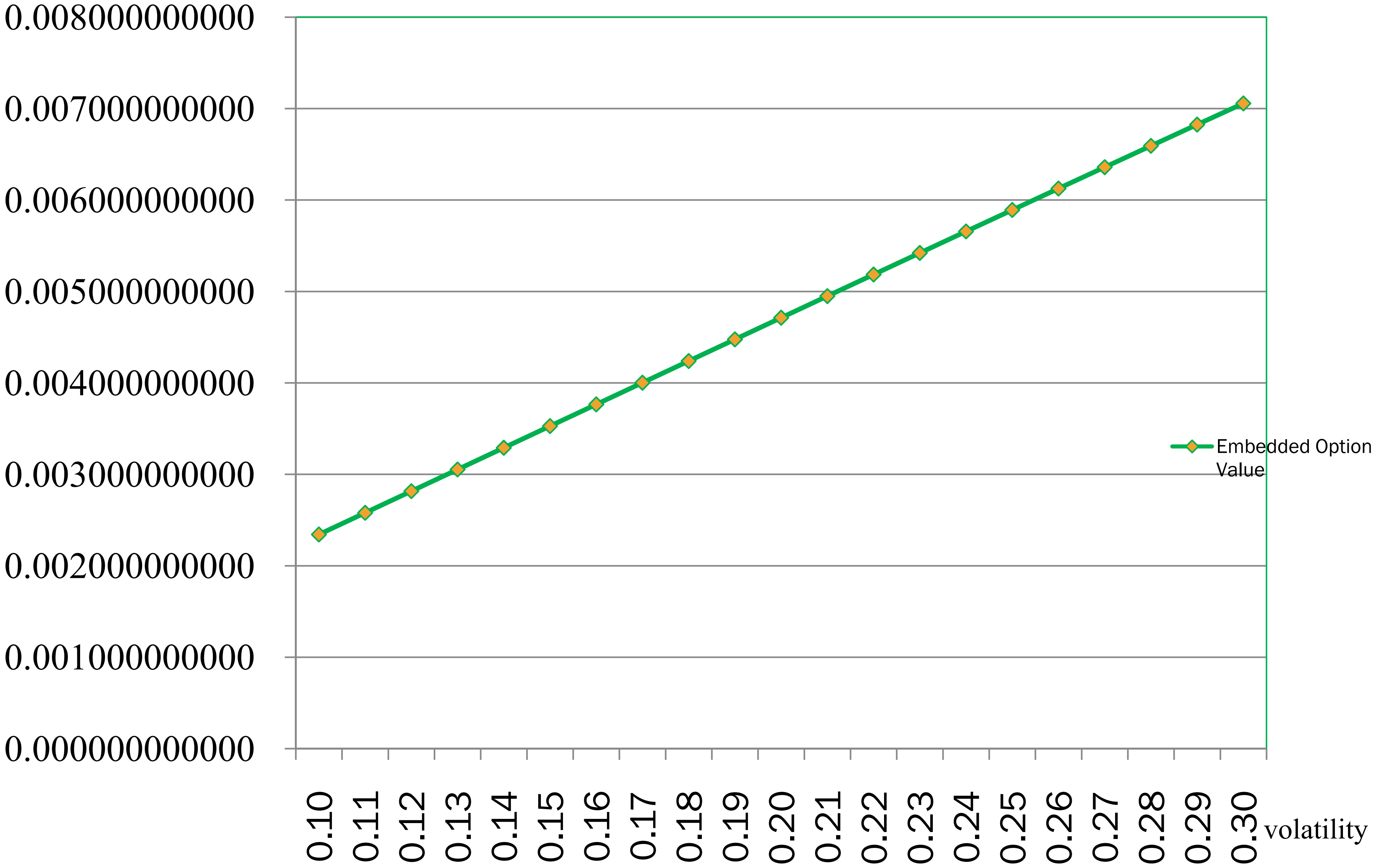


Embedded Option Value
0.002500000000
0.002000000000
0.001500000000
0.001000000000
0.000500000000
0.000000000000
0.952380952
0.952580952
0.952780952
0.952980952
0.953180952
0.953380952
0.953580952
0.953780952
0.953980952
0.954180952
0.954380952
0.954580952
0.954780952
0.954980952
0.955180952
0.955380952
0.955580952
0.955780952
strike price
Embedded Option Value

SENSITIVE ANALYSIS – VOLATILITY

- We increase 1% of the volatility from 10% until reach 30% to see the impact of volatility on the embedded option value
- We find that when the volatility increases, the embedded option value increases
- It is consistent with the inference for the Greek letter *vega* when the underlying asset's volatility increases the option value increases, too.

Embedded Option Value



THE VALUATION OF CAPLETS IN LMM

- The payoff function of a caplet at time T_{i+1} is

$$A \times \delta \times \max(f(T_i; T_i, T_{i+1}) - K, 0)$$

- It is a caplet on the spot rate observed at time T_i with payoff occurring at time T_{i+1}
- The cap is a portfolio consisted of n such call options which the underlying is known as caplet

THE THEROETICAL VALUE OF CAPLET

- We use the Black's formula for the caplet to get the theoretical value of caplet and rewrite as

$$caplet_i(t) = A \times \delta_i \times P(t, T_{i+1})[f(t; T_i, T_{i+1})N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(f(t; T_i, T_{i+1})/K) + \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$

$$d_2 = \frac{\ln(f(t; T_i, T_{i+1})/K) - \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$

NUMERICAL METHOD OF CAPLET

- We use the payoff function to compute the price in lattice method
- To get the payoff function at time T_{i+1}, we have to know the evolution of the forward rate $f(0;T_i,T_{i+1})$ at time T_i
- We construct the binomial tree of $f(0;T_i,T_{i+1})$ and known the $(f(T_i;T_i,T_{i+1})_r - K)^+$, $r = 0, 1, \ldots, i$.
- Calculating the expectation of the payoff at time T_{i+1} and then multiple the ZCB of $P(t,T_{i+1})$ to get the caplet value at time t.

EXAMPLE

- Assume $\delta = A = 1,\ \sigma = 10\%,\ K = 5\%$, forward curve is flat equal to 5% and $n_i = 25,\ i = 1,\ldots,10$

- We have the one period caplet at time 0 $caplet_1(0)$

$$
\begin{aligned}
caplet_1(0) &= A \times \delta \times P(0,2) \times E[\max(f(1;1,2) - K, 0)] \\
&= 1 \cdot 1 \cdot P(0,2) \cdot 0.0020112666 \\
&= 0.90702948 \cdot 0.0020112666 \\
&= 0.0018242781
\end{aligned}
$$

NUMERICAL RESULT

- Besides the relative difference, we use the RMSE to see the difference between the lattice value and Black's model for the whole maturity
- Table 2 is the results for different maturity caplets
- We find that when we increase the stage between periods, the difference between lattice method and Black's model is close

TABLE 2 VOLATILITY IS 10% AND STAGE FOR EVERY PERIOD IS 25

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.0018085085	0.0018242781	0.0000157696	0.8719669656
2	0.0024348117	0.0024407546	0.0000059429	0.2440786919
3	0.0028388399	0.0028374958	-0.0000013441	0.0473484800
4	0.0031206153	0.0031282191	0.0000076038	0.2436631792
5	0.0033214311	0.0033204098	-0.0000010214	0.0307505629
6	0.0034637453	0.0034664184	0.0000026731	0.0771737036
7	0.0035616356	0.0035658574	0.0000042219	0.1185369137
8	0.0036247299	0.0036200240	-0.0000047059	0.1298286011
9	0.0036600091	0.0036633313	0.0000033221	0.0907678678
10	0.0036727489	0.0036743568	0.0000016079	0.0437804213
		RMSE	0.0000063671	

1. Caplet assume $\delta = 1$ and stage 25
2. Assume volatility is 10%, the forward curve is flat 5%

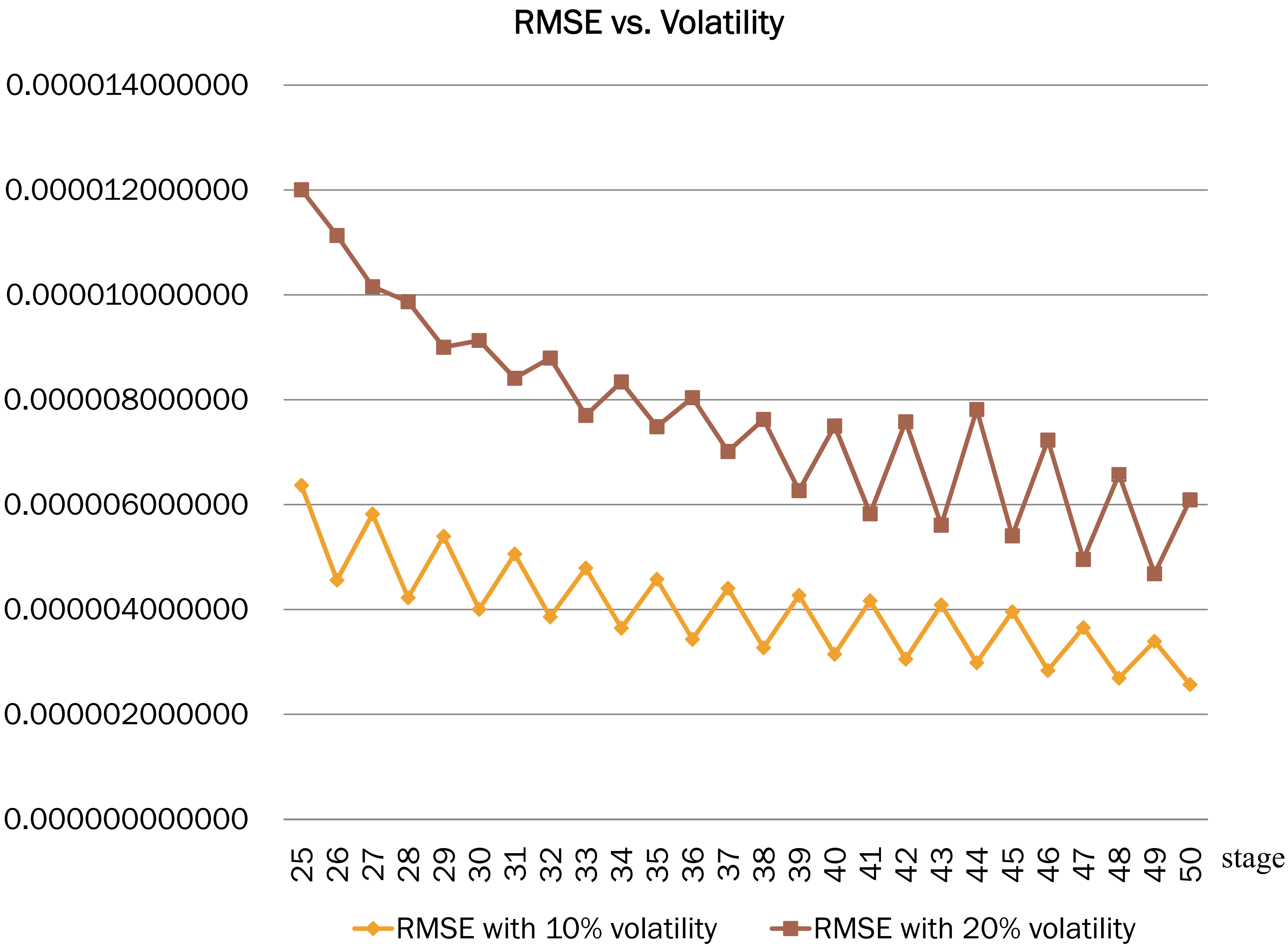
TABLE 2 VOLATILITY IS 10% AND STAGE FOR EVERY PERIOD IS 50

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.0018085085	0.0018099405	0.0000014320	0.0791823392
2	0.0024348117	0.0024397802	0.0000049685	0.2040608652
3	0.0028388399	0.0028434461	0.0000046061	0.1622542164
4	0.0031206153	0.0031230795	0.0000024643	0.0789673074
5	0.0033214311	0.0033207434	-0.0000006878	0.0207066243
6	0.0034637453	0.0034620815	-0.0000016638	0.0480341136
7	0.0035616356	0.0035626664	0.0000010308	0.0289416315
8	0.0036247299	0.0036267433	0.0000020134	0.0555455781
9	0.0036600091	0.0036617883	0.0000017792	0.0486107386
10	0.0036727489	0.0036734197	0.0000006708	0.0182649876
		RMSE	0.0000025690	

1. Caplet assume $\delta = 1$ and stage 50
2. Assume volatility is 10%, the forward curve is flat 5%

CONVERGENCE BEHAVIOR VS. VOLATLITY

- We change the volatility from 10% to 20% to see the convergence behavior of RMSE at different volatility level
- We find that when the volatility at low level, the convergent rate is fast to zero. However, the volatility at high level the convergence behavior is slow


RMSE vs. Volatility
0.000014000000
0.000012000000
0.000010000000
0.000008000000
0.000006000000
0.000004000000
0.000002000000
0.000000000000
25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50
stage
RMSE with 10% volatility
RMSE with 20% volatility

CONCLUSIONS

- Constructing the recombining binomial tree, the payoff of the interest rate derivatives on each node can be obtained
- Comparing to the theoretical value, we find the theoretical value and lattice method is close when we increase the stages.
- The sensitive analysis results of embedded option are consistent with inference of Greek letters

FUTURE WORKS

- Deriving the joint probability between the states of different binomial forward rate trees.
- Adjusting the stages between period by period to fit the strike price to reduce the nonlinearity error
- Trying to change the constant volatility to stochastic volatility to fit the volatility term structure.

The End

THANK YOU FOR YOUR LISTENING

THE RELATIONSHIP BETWEEN DISCRETE TIME FORWARD RATE

Time 0	Time 1	Time 2	Time 3	…
f(0;0,1)				
f(0;1,2)	f(1;1,2)			
f(0;2,3)	f(1;2,3)	f(2;2,3)		
f(0;3,4)	f(1;3,4)	f(2;3,4)	f(3;3,4)	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

(1) drift

(2) volatility

(3) covariance

THE DISCRETE TIME VERSION OF LMM

- The most important results which are under the "risk neutral" measure in the Poon and Stapleton text (2005)

- For a zero-coupon bond price is given by

$$P(t,T_n) = P(t,T_1)E_t(P(T_1,T_n))$$

- The drift of the forward bond price is given by

$$E_t[For(T_1,T_i,T_n)] - For(t,T_i,T_n)$$
$$= -\frac{P(t,T_1)}{P(t,T_n)}\operatorname{cov}_t[For(T_1,T_i,T_n), P(T_1,T_n)]$$

THE MOST IMPORTANT RESULTS IN TEXT

- The drift of *T*-period forward rate is given by

$$E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1}) =$$

$$-\operatorname{cov}_t[f(T_1;T_n,T_{n+1}), \frac{1}{1+y_{T_1}} \times \frac{1}{1+f(T_1;T_2,T_3)} \times \cdots \times \frac{1}{1+f(T_1;T_n,T_{n+1})}]$$

$$\times(1+f(t;T_1,T_2))\cdot(1+f(t;T_2,T_3))\cdot\cdots\cdot(1+f(t;T_n,T_{n+1}))$$

DISCRETE-TIME VERSION OF LMM

- We now apply the results to the LIBOR basis for the FRA and rewrite as follows

$$FRA(T_n) = \frac{A(f(T_n;T_n,T_{n+1}) - K)\cdot\delta_n}{1+\delta_n f(T_n;T_n,T_{n+1})}$$

where $\delta_n = T_{n+1} - T_n$ and we assume all the tenors are same to make the equation brief, that is

$$\delta_1 = \delta_2 = \cdots = \delta_n = \delta$$

DISCRETE-TIME VERSION OF LMM

- We use the above results and similar steps to derive the FRA value at time t to generalize the T_n maturity forward rate

$$E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1}) =$$

$$\frac{-1}{\delta}\mathrm{cov}_t[\delta f(T_1;T_n,T_{n+1}), \frac{1}{1+\delta f(T_1;T_1,T_2)}\cdots\cdots\frac{1}{1+\delta f(T_1;T_n,T_{n+1})}]$$

$$\times(1+\delta f(t;T_1,T_2))\cdot(1+\delta f(t;T_2,T_3))\cdots\cdots(1+\delta f(t;T_n,T_{n+1}))$$

DISCRETE-TIME VERSION OF LMM

- We assume that the forward rate $f(T_1;T_n,T_{n+1})$ is the lognormal for all forward maturities T_n
- We use the approximate result for the covariance term, i.e. for the small change around the value $X=a,\ Y=b$, we have $\mathrm{cov}(X,Y)\approx ab\,\mathrm{cov}(\ln X,\ln Y)$
- Here, we take $a=f(t;T_1,T_2)$ and $b=1/(1+f(t;T_1,T_2))$ to evaluate $\mathrm{cov}_t(f(T_1;T_1,T_2),\frac{1}{1+f(T_1;T_1,T_2)})$, then we have

$$\mathrm{cov}_t(f(T_1;T_1,T_2),\frac{1}{1+f(T_1;T_1,T_2)})=$$

$$f(t;T_1,T_2)(\frac{1}{1+f(t;T_1,T_2)})\,\mathrm{cov}_t(\ln y(T_1,T_2),\ln\frac{1}{1+y(T_1,T_2)})$$

DISCRETE-TIME VERSION OF LMM

- Using the property of logarithms to express the drift of T_n-maturity forward rate as the sum of a series of covariance term
- Using Stein's lemma to make covariance terms in a recognizable form and can be express as

$$E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1}) =$$

$$f(t;T_n,T_{n+1}) \times \frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)} \cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}), \ln f(T_1;T_1,T_2)]$$

$$+\cdots$$

$$+ f(t;T_n,T_{n+1}) \times \frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})} \cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}), \ln f(T_1;T_n,T_{n+1})]$$

DISCRETE-TIME VERSION OF LMM

- Define

$$\text{cov}_t[\ln f(T_1;T_i,T_{i+1}), \ln f(T_1;T_n,T_{n+1})] \equiv \tilde{\sigma}_{i,n} \qquad i = 1,\ 2, \cdots, n$$

- Then, we can rewrite the drift of the forward LIBOR rate as

$$\frac{E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1})}{f(t;T_n,T_{n+1})} = \frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)} \cdot \tilde{\sigma}_{1,n} + \frac{\delta f(t;T_2,T_3)}{1+\delta f(t;T_2,T_3)} \cdot \tilde{\sigma}_{2,n} + \cdots + \frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})} \cdot \tilde{\sigma}_{n,n}$$