Stochastic Calculus For Finance - volume 2

- **Section 4.5.4 Solution to the Black-Scholes-Merton Equation**
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Section 4.5.4 Solution to the Black-Scholes-Merton Equation

- $c_t(t,x) = rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)\ ,\ \forall t \in [0,T)\ ,\ x \geq 0 \quad \cdots \quad (4.5.14)$
- We want the BSM equation to hold $\forall x \geq 0$ and $t \in [0,T)$ so that (4.5.14) will hold regardless of which of its possible paths the stock follows.
- We do not need (4.5.14) to hold at $t = T$, although we need the function $c(t,x)$ to be continuous at $t = T$.
- For such an equation, in addition to the terminal condition (4.5.15) , one needs boundary conditions at $x = 0$ and $x = \infty$ in order to determine the solution.

• Section 4.5.4 Solution to the Black-Scholes-Merton Equation

- $c_t(t,x) = rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$, $\forall t \in [0,T)$, $x \geq 0$ $\cdots$ (4.5.14)

- The boundary conditions at $x = 0$ is obtained by substituting $x = 0$ into (4.5.14) , which the becomes $c_t(t,0) = rc(t,0)$. $\cdots$ (4.5.16)

- 通解為 $c(t,0) = Ce^{rt}c(t,0)$ ，其中 C 為常數項

- 又 $c(T,0) = max\{0 - K, 0\} = 0$ $\Longrightarrow$ $C = 0$ 帶回通解得 $c(t,0) = 0$, $\forall t \in [0,T]$

- We see that $c(0,0) = 0$

- The boundary condition at $x = 0$ $\Longrightarrow$ $c(t,0) = 0$, $\forall t \in [0,T]$ $\cdots$ (4.5.17)

•Section 4.5.4 Solution to the Black-Scholes-Merton Equation

- The boundary condition at $x \to \infty$ for the European call is

  $lim_{x\to\infty}\left[c(t,x) - (x - e^{-r(T-t)}K)\right] = 0 \ ,\forall t \in [0,T] \quad \cdots \ (4.5.18)$

- $<$ pf $>$

Case I : $x >>> K$ (deep in the money)

$$c(t,x) \approx x - Ke^{-r(T-t)}$$

Case II : $x \to \infty$, call 執行的機率 $\to 1$

$$\Longrightarrow \ N(d_+) \to 1 \ , N(d_-) \to 1 \text{ 代入 } c(t,x) = xN(d_+) - Ke^{-r(T-t)}N(d_-)$$

得 $c(t,x) \approx x - Ke^{-r(T-t)}$

$$\therefore \ lim_{x\to\infty}\left[c(t,x) - (x - e^{-r(T-t)}K)\right] = 0 \ ,\forall t \in [0,T]$$

•Section 4.5.4 Solution to the Black-Scholes-Merton Equation

- The solution to the BSM equation (4.5.14) with terminal condition (4.5.15) and boundary conditions (4.5.17) and (4.5.18) is

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d\_(T-t,x)) \; ,0 \leq \, < T \; , \; x > 0 \quad \cdots \; (4.5.19)$$

where $$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right] \quad \cdots \; (4.5.20)$$

and N is the cdf of normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}}\int_{y}^{\infty} e^{-\frac{z^2}{2}} dz \quad \cdots \; (4.5.21)$$

•Section 4.5.4 Solution to the Black-Scholes-Merton Equation

- We shall sometimes use the notation

  $BSM(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - Ke^{-r\tau}N(d_-(\tau, x)) \quad \cdots \quad (4.5.22)$

- In this formula, τ and x denote the time to expiration and the current stock price, respectively. The parameters $K, r,$ and σ are strike price, the interest rate, and the stock volatility, respectively.

- $lim_{t \to T}\ c(t, x) = (x - K)^+$ ($\because \tau = T - t = 0$)

- $lim_{x \to 0}\ c(t, x) = 0$ ($\because$ $log0$ is not a real number)

•Section 4.5.5 The Greek

- $Delta: c_x(t,x) = N(d_+(T-t,x)) \quad \cdots \quad (4.5.23)$
- < pf >

$$d_+(\tau,x) = \frac{ln(\frac{x}{K}) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d\_(\tau,x) = \frac{ln(\frac{x}{K}) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \frac{ln(\frac{x}{K}) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau} = d_+(\tau,x) - \sigma\sqrt{\tau}$$

$$\Longrightarrow \; d_+(\tau,x) = d\_(\tau,x) + \sigma\sqrt{\tau}$$

•Section 4.5.5 The Greek

$$N'(d_-(\tau,x)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_-(\tau,x))^2}$$

$$N'(d_+(\tau,x)) = N'(d_-(\tau,x) + \sigma\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_-(\tau,x)+\sigma\sqrt{\tau})^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_-^2+2d_-\sigma\sqrt{\tau}+\sigma^2\tau)} = N'(d_-(\tau,x))e^{-\frac{1}{2}(2d_-\sigma\sqrt{\tau}+\sigma^2\tau)}$$

$$\Longrightarrow N'(d_+(\tau,x)) = N'(d_-(\tau,x)) \times \frac{Ke^{-r\tau}}{x} \qquad \therefore xN'(d_+(\tau,x)) = Ke^{-r\tau}N'(d_-(\tau,x))$$

•Section 4.5.5 The Greek

其中 $d_+(\tau, x) = \dfrac{ln(\frac{x}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$

$$\Longrightarrow ln(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)\tau = d_+(\tau, x)\sigma\sqrt{\tau}$$

$$\Longrightarrow \frac{x}{K} + e^{(r+\frac{1}{2}\sigma^2)\tau} = e^{d_+(\tau,x)\sigma\sqrt{\tau}}$$

$$\Longrightarrow x = Ke^{d_+(\tau,x)\sigma\sqrt{\tau}-(r+\frac{1}{2}\sigma^2)\tau} = Ke^{-r\tau}e^{d_+(\tau,x)\sigma\sqrt{\tau}-\frac{1}{2}\sigma^2\tau} = Ke^{-r\tau}e^{(d\_(\tau,x)+\sigma\sqrt{\tau})\sigma\sqrt{\tau}-\frac{1}{2}\sigma^2\tau}$$

$$= Ke^{-r\tau}e^{d\_(\tau,x)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau}$$

$$\Longrightarrow e^{-(d\_(\tau,x)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau)} = \frac{Ke^{-r\tau}}{x} \quad \therefore\ e^{-\frac{1}{2}(2d\_(\tau,x)\sigma\sqrt{\tau}+\sigma^2\tau)} = \frac{Ke^{-r\tau}}{x}$$

•Section 4.5.5 The Greek

又 $$\frac{\partial d_+(\tau,x)}{\partial x}=\frac{1}{x\sigma\sqrt{\tau}}$$

$$\frac{\partial d_-(\tau,x)}{\partial x}=\frac{1}{x\sigma\sqrt{\tau}}$$

$$\Longrightarrow \frac{\partial d_+(\tau,x)}{\partial x}=\frac{\partial d_-(\tau,x)}{\partial x}$$

• Section 4.5.5 The Greek

- $Delta: c_x(t,x) = \frac{\partial C(t,x)}{\partial x}$

$$= N(d_+(T-t,x)) + xN'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial x} - Ke^{-r(T-t)}N'(d_-(T-t,x))\frac{\partial d_-(T-t,x)}{\partial x}$$

$$= N(d_+(T-t,x)) + xN'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial x} - xN'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial x}$$

$$\Longrightarrow c_x(t,x) = N(d_+(T-t,x)) \quad \cdots \quad (4.5.23)$$

- $Delta: c_x(t,x) = \frac{\partial C(t,x)}{\partial x} = N(d_+(T-t,x)) > 0$

• Section 4.5.5 The Greek

- $Theta : c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) \quad \cdots \quad (4.5.24)$

- < pf >

$$d_\pm(T-t,x) = \frac{ln\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{\tau}} = \frac{1}{\sigma}ln\left(\frac{x}{K}\right)(T-t)^{-\frac{1}{2}} + \frac{1}{\sigma}\left(r \pm \frac{1}{2}\sigma^2\right)(T-t)^{\frac{1}{2}}$$

$$\frac{\partial d_\pm(T-t,x)}{\partial t} = \frac{1}{2\sigma}ln\left(\frac{x}{K}\right)(T-t)^{-\frac{3}{2}} - \frac{1}{2\sigma}\left(r \pm \frac{1}{2}\sigma^2\right)(T-t)^{-\frac{1}{2}}$$

$$= \frac{1}{2\sigma}(T-t)^{-\frac{3}{2}}\left[ln\left(\frac{x}{K}\right) - \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right]$$

• Section 4.5.5 The Greek

- $c_t(t,x) = \dfrac{\partial c(t,x)}{\partial t}$

$$= xN'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial t} - \left[rKe^{-r(T-t)}N(d\_(T-t,x)) + Ke^{-r(T-t)}N'(d\_(T-t,x))\frac{\partial d\_(T-t,x)}{\partial t}\right]$$

$$= xN'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial t} - rKe^{-r(T-t)}N(d\_(T-t,x)) - xN'(d_+(T-t,x))\frac{\partial d\_(T-t,x)}{\partial t}$$

$$= xN'(d_+(T-t,x))\left[\frac{\partial d_+(T-t,x)}{\partial t} - \frac{\partial d\_(T-t,x)}{\partial t}\right] - rKe^{-r(T-t)}N(d\_(T-t,x))$$

•Section 4.5.5 The Greek

又 $d_+(T-t,x) - d\_(T-t,x) = \sigma\sqrt{T-t}$

$$\Longrightarrow \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) = \frac{-\sigma}{2\sqrt{T-t}}$$

$$\therefore \ c_t(t,x) = -rKe^{-r(T-t)}N(d\_(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) \quad \cdots \ (4.5.24)$$

- $Theta: c_t(t,x) = \frac{\partial c(t,x)}{\partial t} = -rKe^{-r(T-t)}N(d\_(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) < 0$

•Section 4.5.5 The Greek

- $Gamma : c_{xx}(t,x) = \dfrac{\partial^2 c(t,x)}{\partial x^2}$

$$= \frac{\partial c_x(t,x)}{\partial x} = \frac{\partial}{\partial x}(N(d_+(T-t,x)))$$

$$= N'(d_+(T-t,x))\frac{\partial d_+(T-t,x)}{\partial x} = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x))$$

$$\therefore\ c_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)) \quad \cdots \quad (4.5.25)$$

- $Gamma : c_{xx}(t,x) = \dfrac{\partial^2 c(t,x)}{\partial x^2} = \dfrac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)) > 0$

•Section 4.5.5 The Greek

- The hedging portfolio value is $c(t,x) = xN(d_+) - Ke^{-r(T-t)}N(d_-)$
- $xN(d_+) = xc_x(t,x)$ of this value is invested in stock
- $c(t,x) - xc_x(t,x) = -Ke^{-r(T-t)}N(d_-)$ of the amount invested in the money market
- Hedge a long call should hold $-c_x$ shares of stock and invested $Ke^{-r(T-t)}N(d_-)$ in the money market account.

•Section 4.5.5 The Greek

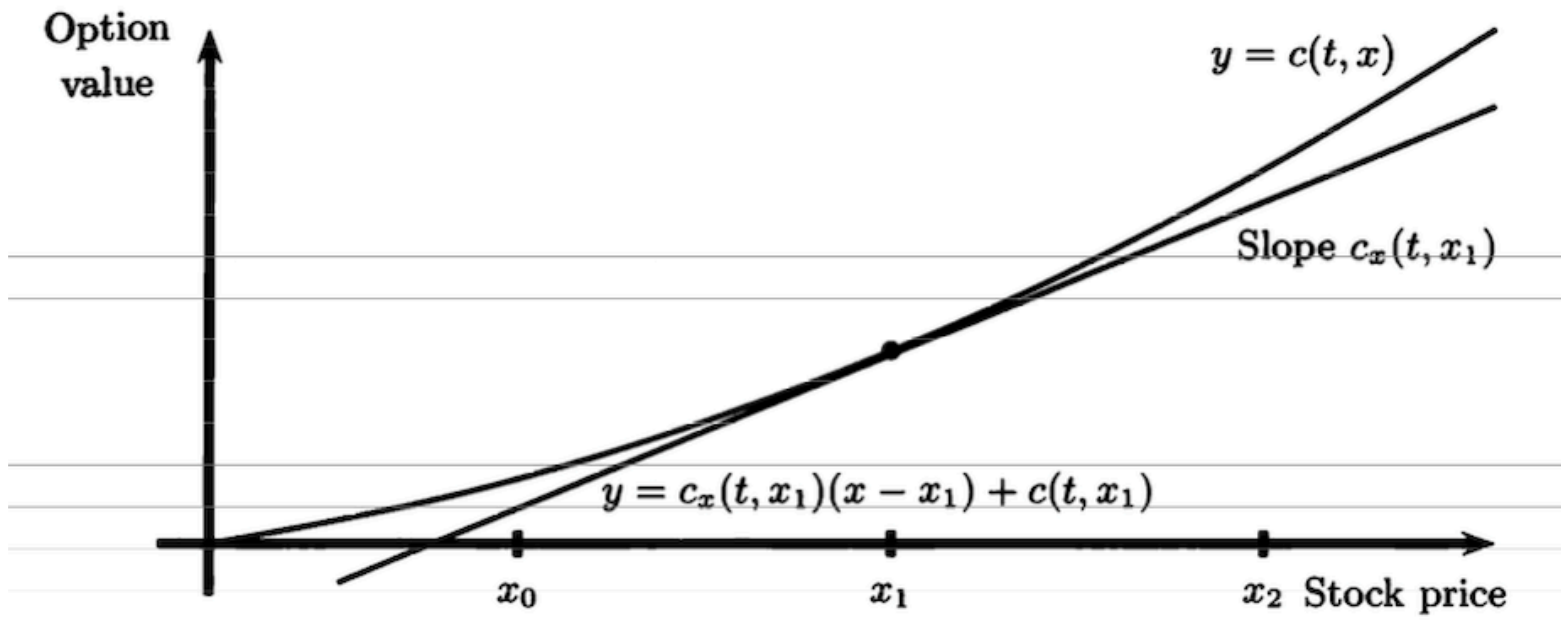

Fig. 4.5.1. Delta-neutral position.

- $c(t, x)$ is increasing and convex in the variable x ($\because c_x' > 0$ and $c_x'' > 0$)
- Take a long position in the option and hedge it
- Buy the option for $c(t, x_1)$, shorting $c_x(t, x_1)$ shares of stock
- Invested the amount $M = x_1 c_x(t, x_1) - c(t, x_1)$ in the money market

•Section 4.5.5 The Greek

- The initial portfolio value $c(t, x_1) - x_1 c_x(t, x_1) + M = 0$
- If the stock price fall to x_0 , the value of the option we hold would fall to $c(t, x_0)$ and the liability due to our short position in stock would decrease to $x_0 c_x(t, x_1)$.
- Total portfolio value

$$c(t, x_0) - x_0 c_x(t, x_1) + M$$

$$= c(t, x_0) - x_0 c_x(t, x_1) + [x_1 c_x(t, x_1) - c(t, x_1)]$$

$= c(t, x_0) - c_x(t, x_1)(x_0 - x_1) - c(t, x_1)$ is the difference at x_0 between the curve y and the straight line y in Figure 4.5.1

•Section 4.5.5 The Greek

- The initial portfolio value $c(t, x_1) - x_1 c_x(t, x_1) + M = 0$
- If the stock price rise to x_2 , the value of the option we hold would rise to $c(t, x_2)$ and the liability due to our short position in stock would increase to $x_2 c_x(t, x_1)$.
- Total portfolio value

$$c(t, x_2) - x_2 c_x(t, x_1) + M$$

$$= c(t, x_2) - x_2 c_x(t, x_1) + [x_1 c_x(t, x_1) - c(t, x_1)]$$

$= c(t, x_2) - c_x(t, x_1)(x_2 - x_1) - c(t, x_1)$ is the difference at x_0 between the curve y and the straight line y in Figure 4.5.1

•Section 4.5.5 The Greek

- The portfolio we have set up is said to be delta-neutral and long gamma.
- A long gamma portfolio is profitable in times of high stock volatility.
- Delta-neutral means that small changes in the stock price result in nearly equal changes in the option price and the value of the short stock position, offsetting each other.
- If the straight line were steeper than the option price curve at the starting point x_1 , then we would be short delta

Section 4.5.5 The Greek

- From the point of view of no-arbitrage pricing, it is irrelevant how likely the stock is to go up or down because a delta-neutral position is a hedge against both possibilities.
- What matters is how much volatility the stock has, for we need to know the amount of profit that can be made from the long gamma position.
- $Vega : \dfrac{\partial c(t,x)}{\partial \sigma} > 0$
- The more volatile stocks offer more opportunity for profit from the portfolio that hedges a long call position with a short stock position.

•Section 4.5.6 Put-Call Parity

- Let $f(t,x)$ denote the value of the forward contract at earlier times $t \in [0,T]$ if the stock price at time t is $S(t) = x$.
- Forward contract : $f(t,x) = x - e^{-r(T-t)}K \quad \cdots \quad (4.5.26)$
- If $t = 0$, 代入 $f(t,S(0)) = S(0) - e^{-rT}K$
- Replicate the value of the forward contract with a portfolio at each time t is $S(t) - e^{-r(T-t)}K$
- Let $S(t) - e^{-r(T-t)}K = 0 \implies S(t) = e^{-r(T-t)}K \implies K = e^{r(T-t)}S(t)$
- $For(t) = e^{r(T-t)}S(t) \quad \cdots \quad (4.5.27)$

Section 4.5.6 Put-Call Parity

- The forward price is not the price (or value) of a forward contract.
- Let $t = 0 \implies For(0) = e^{rT}S(0) \implies K = e^{rT}S(0)$
- Value at time t is $f(t, S(t)) = S(t) - e^{rt}S(0)$

•Section 4.5.6 Put-Call Parity

- $European\ put = (K - S(T))^{+}$ at time T
- For any number x, the equation $x - K = (x - K)^{+} - (K - x)^{+} \quad \cdots \quad (4.5.28)$
- If $x \geq K$, then $(x - K)^{+} - (K - x)^{+} = (x - K) - 0 = x - K.$
- If $x \leq K$, then $(x - K)^{+} - (K - x)^{+} = -(K - x) = x - K.$
- Equation (4.5.28) implies $f(T, S(T)) = c(T, S(T)) - p(T, S(T))$
- A portfolio that is long a call and short a put.

•Section 4.5.6 Put-Call Parity

- $p(t,x) = x(N(d_+(T-t,x)) - 1) - Ke^{-r(T-t)}(N(d\_(T-t,x)) - 1)$

  $= Ke^{-r(T-t)}N(-d\_(T-t,x)) - xN(-d_+(T-t,x)) \quad \cdots \quad (4.5.30)$

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At all previous times : $f(t,x) = c(t,x) - p(t,x),\ x \geq 0,\ 0 \leq t \leq T$

由 (4.5.26) $f(t,x) = x - e^{-r(T-t)}K$ 代入得 $c(t,x) - p(t,x) = x - e^{-r(T-t)}K$

$\Longrightarrow p(t,x) = c(t,x) - x + e^{-r(T-t)}K \quad \cdots \quad (1)$

•Section 4.5.6 Put-Call Parity

又知 (4.5.19) $c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d\_(T-t,x))$, $0 \le \ < T$, $x > 0$

代入 (1) 得 $p(t,x) = \left[xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d\_(T-t,x))\right] - x + e^{-r(T-t)}K$

$$= Ke^{-r(T-t)}\left[1 - N(d\_(T-t,x))\right] - x\left[1 - N(d_+(T-t,x))\right]$$

$$= Ke^{-r(T-t)}N(-d\_(T-t,x)) - xN(-d_+(T-t,x))$$

$\therefore\ p(t,x) = Ke^{-r(T-t)}N(-d\_(T-t,x)) - xN(-d_+(T-t,x))$