Stochastic Calculus For Finance - volume 2

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Section 8.3.1 Price Under Optimal Exercise

Lemma 8.3.4. *The function $v_L(x)$ is given by the formula*

$$v_L(x) = \begin{cases} K - x, & 0 \le x \le L, \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L. \end{cases} \tag{8.3.11}$$

$v_{L_1}(x)$

- lies below the intrinsic value $K - x$ for x between L_1 and L_2.

$v_{L_2}(x)$

- agrees with the intrinsic value for $0 \le x \le L_2$ and follows the indicated curve for $x \ge L_2$.

$v_{L^*}(x)$

- agrees with the intrinsic value for $0 \le x \le L^*$.
- For $x \ge L^*$, the function $v_{L^*}(x)$ is strictly larger than the function $v_{L_2}(x)$, and hence the strategy of exercising the first time the stock price falls to L^* is better than exercising the first time the stock price falls to L_2 .

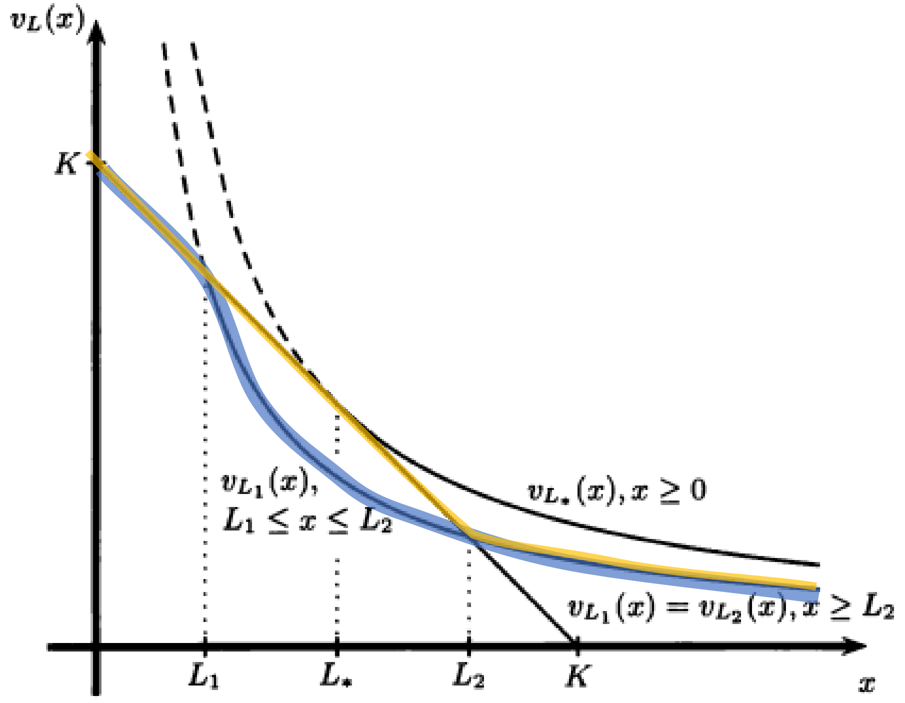

Fig. 8.3.1. $(K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$ for three values of L.

We must first determine the value of L^*. We note that

$$v_L(x) = (K-L)L^{\frac{2r}{\sigma^2}}x^{-\frac{2r}{\sigma^2}} \qquad for\ all\ x \geq L$$

From Figure 8.3.1, we know that L^* is the value of L that maximizes this quantity when we hold x fixed. We thus define $g(L) = (K-L)L^{\frac{2r}{\sigma^2}}$ and seek the value of L that maximizes this function over $L \geq 0$. Because $\frac{2r}{\sigma^2}$ strictly positive, we have $g(0) = 0$ and $\lim_{L\to\infty} g(L) = -\infty$. Moreover,

$$g'(L) = -L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}(K-L)L^{\frac{2r}{\sigma^2}-1} = -L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1} - \frac{2r}{\sigma^2}L^{\frac{2r}{\sigma^2}} = -\frac{2r+\sigma^2}{\sigma^2}L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1}$$

Setting this equal to zero, we solve for

$$-\frac{2r+\sigma^2}{\sigma^2}L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1} = 0$$

$$\frac{2r}{\sigma^2}KL^{\frac{2r}{\sigma^2}-1} = \frac{2r+\sigma^2}{\sigma^2}L^{\frac{2r}{\sigma^2}}$$

$$\frac{\frac{2r}{\sigma^2}K}{\frac{2r+\sigma^2}{\sigma^2}} = \frac{L^{\frac{2r}{\sigma^2}}}{L^{\frac{2r}{\sigma^2}-1}} \quad \Rightarrow \quad L^* = \frac{2r}{2r+\sigma^2}K \qquad ,0 \leq L^* \leq K \qquad (8.3.12)$$

Furthermore,

$$g(L^*) = (K - L^*)L^{*\frac{2r}{\sigma^2}} = \left(K - \frac{2r}{2r+\sigma^2}K\right)\left(\frac{2r}{2r+\sigma^2}K\right)^{\frac{2r}{\sigma^2}}$$

$$= \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}} K^{\frac{2r+\sigma^2}{\sigma^2}}$$

is strictly positive.

Therefore, the graph of $y = g(L)$ must be as shown in Figure 8.3.2, and L^* given by (8.3.12) is the point where $g(L)$ at

$$g'(L) = -\frac{2r+\sigma^2}{\sigma^2} L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2} K L^{\frac{2r}{\sigma^2}-1}$$

$$g''(L) = -\frac{2r+\sigma^2}{\sigma^2}\cdot\frac{2r}{\sigma^2} L^{\frac{2r}{\sigma^2}-1} + \frac{2r}{\sigma^2} K\left(\frac{2r}{\sigma^2}-1\right) L^{\frac{2r}{\sigma^2}-2}$$

$$= -\frac{2r(2r+\sigma^2)}{\sigma^4} L^{\frac{2r}{\sigma^2}-1} + \frac{2r(2r-\sigma^2)}{\sigma^4} K L^{\frac{2r}{\sigma^2}-2} = 0$$

(inflection point)

$$2r(2r-\sigma^2) K L^{\frac{2r}{\sigma^2}-2} = 2r(2r+\sigma^2) L^{\frac{2r}{\sigma^2}-1}$$

$$\frac{2r-\sigma^2}{2r+\sigma^2} K = L < L^*$$

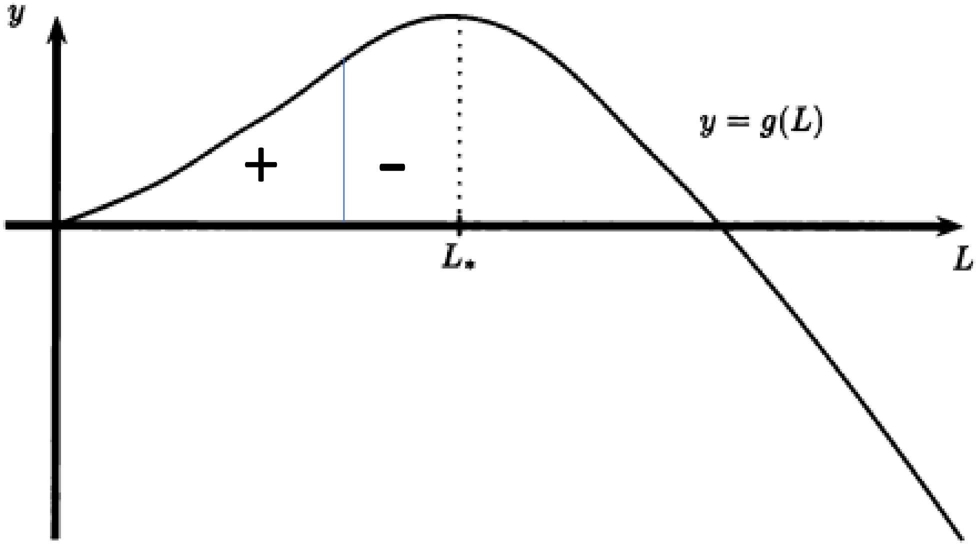

Fig. 8.3.2. Graph of $g(L)$.

Section 8.3.2 Analytical Characterization of the Put Price

We have

$$v_{L^*}(x) = \begin{cases} K - x & 0 \le x \le L^* \\ (K - L^*)\left(\dfrac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} & x \ge L^* \end{cases} \qquad (8.3.13)$$

So that

$$v'_{L^*}(x) = \begin{cases} -1 & 0 \le x \le L^* \\ -(K - L^*)\dfrac{2r}{\sigma^2 x}\left(\dfrac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} & x \ge L^* \end{cases} \qquad (8.3.14)$$

If we evaluate the second line in (8.3.14) at $x = L^*$, we get the right-hand derivative

$$v'_{L^*}(L^* +) = -(K - L^*)\frac{2r}{\sigma^2 L^*} = -\frac{2rK}{\sigma^2 L^*} + \frac{2r}{\sigma^2} = -\frac{2r}{\sigma^2} \cdot \frac{2r + \sigma^2}{2r} + \frac{2r}{\sigma^2} = -1$$

which agrees with the left-hand derivative $v'_{L^*}\ (L^* -) = -1$ provided by the first line in (8.3.14). The derivative of $v_{L^*}\ (x)$ is continuous at $x = L^*$. This is known as *smooth pasting*.

The second derivative of $v(x)$ has a jump at $x = L^*$, and hence is undefined at this point. Indeed,

$$v_{L^*}''(x) = \begin{cases} 0 & , \quad 0 \le x < L^* \\ (K - L^*)\left(\frac{2r(2r+\sigma^2)}{\sigma^4 x^2}\right)\left(\frac{x}{L^*}\right)^{-\frac{2x}{\sigma^2}} & , \quad x > L^* \end{cases} \tag{8.3.15}$$

The left-hand and right-hand second derivatives at $x = L^*$ are $v(L^*-) = 0$ and $v''(L^* +) = (K - L^*)^{\frac{2r(2r+\sigma^2)}{\sigma^4 L^{*2}}} > 0$.

For $x > L^*$, we can verify by direct computation that

$$\begin{aligned} & r\, v_{L^*}(x) - rxv'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x) \\ &= r(K - L^*)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} - rx\left[-(K - L^*)\frac{2r}{\sigma^2 x}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}\right] - \frac{1}{2}\sigma^2 x^2 (K - L^*)\left(\frac{2r(2r+\sigma^2)}{\sigma^4 x^2}\right)\left(\frac{x}{L^*}\right)^{-\frac{2x}{\sigma^2}} \\ &= (K - L^*)\left(\frac{x}{L^*}\right)^{-\frac{2x}{\sigma^2}}\left[r + r\frac{2r}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right] = 0 \end{aligned} \tag{8.3.16}$$

On the other hand, for $0 \le x < L^*$,

$$r\, v_{L^*}(x) - rxv'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x) = r(K - x) - rx(-1) - \frac{1}{2}\sigma^2 x^2 0 = rK$$

In particular, we see that $v_{L^*}(x)$ satisfies the so-called *linear complementarity conditions*

$$v(x) \geq (K - x)^+ \qquad for\ all\ x \geq 0, \tag{8.3.18}$$

$$r\, v_{L^*}(x) - rxv'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x) \geq 0 \qquad for\ all\ x \geq 0, \text{ and} \tag{8.3.19}$$

for each $x \geq 0$, equality holds in either (8.3.18) or (8.3.19). (8.3.20)

The point L^* is slightly problematical in (8.3.19) since $v''_{L^*}(x)$ is undefined. However, if we replace $v''_{L^*}(L^*)$ in (8.3.19) by either $v''_{L^*}(L^* -)$ or $v''_{L^*}(L^* +)$, the inequality holds.

The linear complementarity conditions (8.3.18)-(8.3.20) determine the function $v_{L^*}(x)$. More precisely, the function $v_{L^*}(x)$ given by (8.3.13) is the only bounded continuous function having a continuous derivative that satisfies these conditions.

Section 8.3.3 Probabilistic Characterization of the Put Price

Theorem 8.3.5. Let $S(t)$ be the stock price given by (8.3.1) and let τ_{L^*} be given by (8.3.9) with $L = L^*$. Then $e^{-rt}v_{L^*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(t\wedge\tau_{L^*})}v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

Proof Fortunately, the Itô–Doeblin formula applies to functions whose second derivatives have jumps, provided the first derivative is continuous. We may thus compute

$$d\left[e^{-rt}v_{L^*}\big(S(t)\big)\right] = e^{-rt}\left[-rv_{L^*}\big(S(t)\big)dt + v'_{L^*}\big(S(t)\big)dS(t) + \frac{1}{2}\, v''_{L^*}(S(t))dS(t)dS(t)\right]$$

$$= e^{-rt}\left[-rv_{L^*}\big(S(t)\big) + rS(t)v'_{L^*}\big(S(t)\big) + \frac{1}{2}\sigma^2S^2(t)v''_{L^*}\big(S(t)\big)\right]dt + e^{-rt}\sigma S(t)v'_{L^*}\big(S(t)\big)d\widetilde{W}(t).$$

Because of (8.3.16) and (8.3.17), the dt term in this expression is either 0 or $-rK$, depending on whether $S(t) > L^*$ or $S(t) < L^*$. If $S(t) = L^*$, $v''_{L^*}\big(S(t)\big)$ is undefined, but the probability $S(t) = L^*$ is zero so this does not matter. We thus have

$$d\left[e^{-rt}v_{L^*}\big(S(t)\big)\right] = -e^{-rt}rK\mathbb{I}_{\{S(t)<L^*\}}dt + e^{-rt}\sigma S(t)v'_{L^*}\big(S(t)\big)d\widetilde{W}(t). \qquad (8.3.21)$$

Because the dt term in (8.3.21) is less than or equal to zero, $e^{-rt}v_{L^*}\big(S(t)\big)$ is a supermartingale; when $S(t) < L^*$ it has a downward tendency. If the initial stock price is above L^* , then prior to the time τ_{L^*}, when the stock price first reaches L^* , the dt term in (8.3.21) is zero and hence $e^{-r(t\wedge\tau_{L^*})}v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale. Indeed, integration of (8.3.21) yields

$S(t) > L^*$

$$e^{-r(t\wedge\tau_{L^*})}v_{L^*}\big(S(t\wedge\tau_{L^*})\big) = v_{L^*}(0) + \int_0^{t\wedge\tau_{L^*}} e^{-ru}\sigma S(u)v'_{L^*}\big(S(u)\big)d\widetilde{W}(u).$$

Itô integrals are martingales, and hence the Itô integral above stopped at the stopping time τ_{L^*} , is a martingale.

Corollary 8.3.6. Recall that $\mathcal{T}$ is the set of all stopping times, not just those of the form (8.3.9). We have

$$v_{L^*}(x) = \max_{\tau \in \mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right],$$

where $x = S(0)$ is the initial stock price. In other words, $v_{L^*}(x)$is the *perpetual American put price.*

Proof

Because $e^{-rt} v_{L^*}\big(S(t)\big)$is a supermartingale under $\widetilde{\mathbb{P}}$,we have from Theorem 8.2.4 (optional sampling) that, for every stopping time $\tau \in \mathcal{T}$,

$$v_{L^*}(x) = v_{L^*}\big(S(0)\big) \geq \widetilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L^*}\big(S(t\wedge\tau)\big)\right]. \qquad (8.3.22)$$

Because $v_{L^*}\big(S(t\wedge\tau)\big)$is bounded, we may let $t \to \infty$ in (8.3.22), using the Dominated Convergence Theorem, Theorem 1.4.9, to conclude that

$v(x) \geq (K - x)^+ \quad for\ all\ x \geq 0,$

$$v_{L^*}(x) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau} v_{L^*}\big(S(\tau)\big)\right] \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right],$$

where we have gotten the last inequality from (8.3.18). Because this inequality holds for every $\tau \in \mathcal{T}$, we have

$$v_{L^*}(x) \geq \max_{\tau \in \mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right].$$

Proof

(Continued)

On the other hand, if we replace τ by τ_{L_*} ,we obtain equality in (8.3.22) because $e^{-r(t\wedge\tau_{L^*})}v\big(S(t\wedge\tau_{L^*})\big)$ is a martingale under $\widetilde{\mathbb{P}}$. Letting $t\to\infty$ and using the Dominated Convergence Theorem, we obtain

$$v_{L^*}(x)=\widetilde{\mathbb{E}}\big[e^{-r\tau_{L^*}}v_{L^*}\big(S(\tau_{L^*})\big)\big].$$

Since

$$e^{-r\tau_{L^*}}v_{L^*}\big(S(\tau_{L^*})\big)=e^{-r\tau_{L^*}}v_{L^*}(L^*)=e^{-r\tau_{L^*}}(K-L^*)=e^{-r\tau_{L^*}}\big(K-S(\tau_{L^*})\big),$$

if $\tau_{L^*}<\infty$ (and is interpreted to be zero if $\tau_{L^*}=\infty$) ,we see that

$$v_{L^*}(x)=\widetilde{\mathbb{E}}[e^{-r\tau_{L^*}}(K-L^*)]=\widetilde{\mathbb{E}}\big[e^{-r\tau_{L^*}}\big(K-S(\tau_{L^*})\big)\big]. \qquad (8.3.23)$$

It follows that $v_{L_*}(x)\leq\max\limits_{\tau\in\mathcal{T}}\widetilde{\mathbb{E}}\big[e^{-r\tau}\big(K-S(\tau_{L^*})\big)\big]$.

Corollary 8.3.7. Consider an agent with initial capital $X(0) = v_{L^*}(S(0))$,the initial perpetual American put price. Suppose this agent uses the portfolio process

$$\Delta(t) = v'_{L^*}(S(t))$$

and consumes cash at rate

$$C(t) = rK\mathbb{I}_{\{S(t)<L^*\}},$$

(i.e., consumes cash at rate rK whenever $S(t) < L^*$) . Then the value $X(t)$ of the agent's portfolio agrees with the option price $v_{L^*}(S(t))$ for all times t until the option is exercised. In particular, $X(t) \geq (K - S(t))^+$ for all t until the option is exercised, so the agent can pay off a short option position regardless of when the option is exercised.

Proof

$$d[e^{-rt}v_{L^*}(S(t))] = -e^{-rt}rK\mathbb{I}_{\{S(t)<L^*\}}dt + e^{-rt}\sigma S(t)v'_{L^*}(S(t))d\widetilde{W}(t). \quad (8.3.21)$$

The differential of the agent's portfolio value process is

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt - C(t)\,dt,$$

so the differential of the discounted portfolio value process is

$$\begin{aligned} d\big(e^{-rt}X(t)\big) &= e^{-rt}\big(-rX(t)\,dt + dX(t)\big) \\ &= e^{-rt}(\Delta(t)\,dS(t) - r\Delta(t)S(t)\,dt - C(t)\,dt) \\ &= e^{-rt}\big(\Delta(t)\sigma S(t)\,d\widetilde{W}(t) - C(t)\,dt\big). \end{aligned} \quad (8.3.24)$$

Substituting $\Delta(t) = v'_{L^*}(S(t))$and $C(t) = rK\mathbb{I}_{\{S(t)<L^*\}}$ into (8.3.24) and comparing it to (8.3.21), we see that

$$d\big(e^{-rt}X(t)\big) = d\left(e^{-rt}v_{L^*}(S(t))\right).$$

Integrating both sides of this equation and using the initial equality $X(0) = v_{L^*}(S(0))$,we obtain $X(t) = v_{L^*}(S(t))$ for all t prior to exercise.