Stochastic Calculus For Finance - volume 2

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Section 7.4.1 Floating Strike Lookback Option

An option whose payoff is based on the maximum that the underlying asset price attains over some interval of time prior to expiration is called a lookback option.

We begin with a geometric Brownian motion asset price, which may be written as in (7.3.1) as

$$S(t) = S(0)e^{\sigma\widehat{W}(t)} \qquad (7.4.1)$$

$$S(t) = S(0)\exp\left((r - \tfrac{1}{2}\sigma^2)t + \sigma\widetilde{W}(t)\right).$$

where, as in Subsection 7.3.1, $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$ and $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

With

$$\widehat{M}(t) = \max_{0\le u\le t} \widehat{W}(u), 0 \le t \le T, \qquad (7.4.2)$$

we may write the maximum of the asset price up to time t as

$$Y(t) = \max_{0\le u\le t} S(u) = S(0)e^{\sigma\widehat{M}(t)} \qquad (7.4.3)$$

The lookback option considered in this section pays off

$$V(T)=Y(T)-S(T) \qquad (7.4.4)$$

at expiration time T. This payoff is nonnegative because $Y(T) \ge S(T)$

Let $t \in [0, \mathrm{T}]$ be given. At time t, the risk-neutral price of the lookback option is

$$V(t) = \mathbb{E}^{Q}\left[e^{-r(T-t)}\big(Y(T) - S(T)\big)\Big|\mathcal{F}(t)\right]$$

Because the pair of processes $(S(t), Y(t))$ has the Markov property, there must exist a function $v(t, x, y)$ such that

$$V(t) = v(t, S(t), Y(t))$$

Section 7.4.2 Black–Scholes–Merton Equation

Theorem 7.4.1. Let $v(t,x,y)$ denote the price at time *t* of the floating strike lookback option under the assumption that $S(t) = x$ and $Y(t) = y$. Then $v(t,x,y)$ satisfies the *Black-Scholes-Merton partial differential equation*

$$v_t(t,x,y) + rxv_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y) \quad (7.4.6)$$

in the region $\{(t,x,y);\ 0 \le t < T, 0 \le x \le y\}$ and satisfies the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}y, \qquad 0 \le t \le T, y \ge 0 \qquad (7.4.7)$$

$$v_y(t,y,y) = 0, \qquad 0 \le t \le T, y > 0 \qquad (7.4.8)$$

$$v(T,x,y) = y - x, \qquad 0 \le x \le y \qquad (7.4.9)$$

Iterated conditioning implies that $e^{-rt}V(t) = e^{-rt}v(t, S(t), Y(t))$, where $V(t)$ is given by (7.4.5), is a martingale under $\widetilde{\mathbb{P}}$. We compute its differential and set the dt term equal to zero to obtain (7.4.6). However, when we do this, the term $dY(t)$ appears. This is different from the term $dS(t)$, because $S(t)$ has nonzero quadratic variation, whereas $Y(t)$ has zero quadratic variation. This is because $Y(t)$ is continuous and nondecreasing in t. Let $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0, T]$.

Then

$$\begin{aligned}\sum_{j=1}^{m}\left(Y(t_j) - Y(t_{j-1})\right)^2 &\leq \max_{j=1,\dots,m}\left(Y(t_j) - Y(t_{j-1})\right)\sum_{j=1}^{m}\left(Y(t_j) - Y(t_{j-1})\right) \\ &= \max_{j=1,\dots,m}\left(Y(t_j) - Y(t_{j-1})\right)\cdot\left(Y(T) - Y(0)\right),\end{aligned} \tag{7.4.10}$$

and $\max_{j=1,\dots,m}\left(Y(t_j) - Y(t_{j-1})\right)$ has limit zero as $\max_{j=1,\dots,m}(t_j - t_{j-1})$ goes to zero because $Y(t)$ is continuous. We conclude that $Y(t)$ accumulates zero quadratic variation on $[0, T]$, a fact we record by writing

$$dY(t)dY(t) = 0. \tag{7.4.11}$$

On the other hand, $dY(t)$ is not a dt term: there is no process $\Theta(t)$ such that

$$dY(t) = \Theta(t)dt.$$

In other words, we cannot write $Y(t)$as

$$Y(t) = Y(0) + \int_0^t \Theta\,(u)\,du. \qquad (7.4.12)$$

if (7.4.12) were to hold, we would need to have $\Theta(u) = 0$ for Lebesgue almost every u in $[0, T]$ .This would result in $Y(t) = Y(0)$ for $0 \leq t \leq T$. But in fact $Y(t) > Y(0)$ for all $t > 0$. We conclude that $Y(t)$ cannot be represented in the form (7.4.12); $dY(t)$ is not a dt term.

The paths of $Y(t)$ increase over time, but they do so on a set of times having zero Lebesgue measure. Each time interval $[0, T]$ contains a sequence of subintervals whose lengths sum to T, and on each of these subintervals, $Y(t)$ is constant. The particular subintervals depend on the path, but regardless of the path, the lengths of these subintervals sum to T. A similar situation is described in Appendix A, Section A.3. In the case discussed there, $T = 1$ and the subintervals are explicitly exhibited. Their union is the Cantor set. It is verified that although the lengths of these subintervals sum to 1, there are uncountably many points not contained in these intervals. The function $F(x)$ described in Section A.3 increases, but only on the complement of the Cantor set. Furthermore, $F(x)$is continuous. Functions of this kind are said to be singularly continuous.

Fortunately, we can work with the differential of $Y(t)$ .We have already argued that

$$dY(t)\,dY(t) = 0.$$

Similarly, we have

$$dY(t)\,dS(t) = 0. \qquad (7.4.13)$$

PROOF OF THEOREM 7.4.1: We use the Ito-Doeblin formula and (7.4.11) and (7.4.13) to differentiate the martingale $e^{-rt}v(t,S(t),Y(t))$ to obtain

$$
\begin{aligned}
& d\Big(e^{-rt}v\big(t,S(t),Y(t)\big)\Big) \\
&= e^{-rt}\Big[-rv\big(t,S(t),Y(t)\big)dt + v_t\big(t,S(t),Y(t)\big)dt + v_x\big(t,S(t),Y(t)\big)dS(t) \\
&\qquad + \frac{1}{2}v_{xx}\big(t,S(t),Y(t)\big)dS(t)dS(t) + v_y\big(t,S(t),Y(t)\big)dY(t)\Big] \\
&= e^{-rt}\Big[-rv\big(t,S(t),Y(t)\big) + v_t\big(t,S(t),Y(t)\big) + rS(t)v_x\big(t,S(t),Y(t)\big) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(t)v_{xx}\big(t,S(t),Y(t)\big)\Big]dt \\
&\quad + e^{-rt}\sigma S(t)v_x\big(t,S(t),Y(t)\big)d\widetilde{W}(t) + e^{-rt}v_y\big(t,S(t),Y(t)\big)dY(t).
\end{aligned}
\tag{7.4.14}
$$

In order to have a martingale, the dt term must be zero, and this gives us the Black–Scholes–Merton equation (7.4.6). The new feature is that the term

$$e^{-rt}v_y(t, S(t), Y(t))\, dY(t)$$

must also be zero. It cannot be canceled by the dt term nor by the $dW(t)$ term because it is fundamentally different from both of these terms. The $dY(t)$ term is naturally zero on the "flat spots" of $Y(t)$ i.e., when $S(t) < Y(t)$.(However, at the times when $Y(t)$ increases, which are the times when $S(t) = Y(t)$,the term

$$e^{-rt}v_y(t, S(t), Y(t))\, dY(t)$$

$$v_y(t, y, y) = 0, \qquad 0 \le t \le T, y > 0 \qquad (7.4.8)$$

must be zero because $dY(t)$ is "positive." This gives us the boundary condition (7.4.8).

$$v(T, x, y) = y - x, \qquad 0 \le x \le y \qquad (7.4.9)$$

The boundary condition (7.4.9) is the payoff of the option. If at any time t we have $S(t) = 0$,then we will have $S(T) = 0$ .Furthermore, Y will be constant on $[t, T]$; if $Y(t) = y$,then $Y(T) = y$,and the price of the option at time t is this value discounted from T back to t. This gives us the boundary condition (7.4.7).

$$v(t, 0, y) = e^{-r(T-t)}y, \qquad 0 \le t \le T, y \ge 0 \qquad (7.4.7)$$

Remark 7.4.2. The proof of Theorem 7.4.1 shows that

$$d\left(e^{-rt}v(t,S(t),Y(t))\right) = e^{-rt}\sigma S(t)\, v_x(t,S(t),Y(t))\; d\widetilde{W}(t).$$

Just as in **Remark 7.3.3**, this equation implies that the delta-hedging formula (7.3.15) works. In contrast to the situation in Remark 7.3.3, here the function $v(t,x,y)$ is continuous and we have no problems with large delta and gamma values.

Section 7.4. 3 Reduction of Dimension

The price of the floating strike lookback option has a linear scaling property:

$$v(t, \lambda x, \lambda y) = \lambda v(t, x, y) \text{ } for \text{ } all \text{ } \lambda > 0. \quad (7.4.15)$$

This is because scaling both $S(t)$ and $Y(t)$ by the same positive constant at a time t prior to expiration results in the payoff $Y(T) - S(T)$ being scaled by the same constant. In particular, if we know the function of two variables

$$u(t, z) = v(t, z, 1), 0 \leq t \leq T, 0 \leq z \leq 1, \quad (7.4.16)$$

then we can easily determine the function of three variables $v(t, x, y)$ by the formula

$$v(t, x, y) = y \cdot v\left(t, \frac{x}{y}, 1\right) = y \cdot u\left(t, \frac{x}{y}\right), 0 \leq t \leq T, 0 \leq x \leq y, y > 0. \quad (7.4.17)$$

From (7.4.17), we can compute the partial derivatives:

$$v_t(t,x,y) = y \cdot u_t\left(t,\frac{x}{y}\right)$$

$$v_x(t,x,y) = y \cdot\ u_z\left(t,\frac{x}{y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = u_z\left(t,\frac{x}{y}\right),$$

$$v(t,x,y) = y \cdot u\left(t,\frac{x}{y}\right)$$

$$v_{xx}(t,x,y) = u_{zz}\left(t,\frac{x}{y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{y} \cdot u_{zz}\left(t,\frac{x}{y}\right),$$

$$v_y(t,x,y) = u\left(t,\frac{x}{y}\right) + y \cdot u_z\left(t,\frac{x}{y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = u\left(t,\frac{x}{y}\right) - \frac{x}{y} \cdot u_z\left(t,\frac{x}{y}\right)$$

Substitution into the Black-Scholes-Merton equation (7.4.6) yields

$$0 = -rv(t,x,y) + v_t(t,x,y) + rx\, v_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y)$$

$$= y\left[-r \cdot u\left(t,\frac{x}{y}\right) + u_t\left(t,\frac{x}{y}\right) + r \cdot \left(\frac{x}{y}\right) \cdot u_z\left(t,\frac{x}{y}\right) + \frac{1}{2}\sigma^2 \left(\frac{x}{y}\right)^2 \cdot u_{zz}\left(t,\frac{x}{y}\right)\right]$$

Canceling y and making the change of variable $z = \frac{x}{y}$, we see that $u(t,z)$ satisfies the Black-Scholes-Merton equation

$$u_t(t,z) + rz\,u_z(t,z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = r\,u(t,z) \quad (7.4.18)$$

Boundary conditions for $u(t,z)$ can be obtained from the boundary conditions (7.4.7)- (7.4.9) for $v(t,x,y)$. In particular, $e^{-r(T-t)}y = v(t,0,y) = y\,u(t,0)$ implies

$$u(t,0) = e^{-r(T-t)}, 0 \leq t \leq T. \quad (7.4.19)$$

Furthermore, $0 = v_y(t,y,y) = u(t,1) - u_z(t,1)$ implies

$$u(t,1) = u_z(t,1), 0 \leq t < T. \quad (7.4.20)$$

Finally, $y - x = v(T,x,y) = y\,u\left(T,\frac{x}{y}\right)$ implies

$$u(T,z) = 1 - z, 0 \leq z \leq 1. \quad (7.4.21)$$

Equation (7.4.18) and the boundary conditions (7.4.19)–(7.4.21) uniquely determine the function $u(t,z)$. As a consequence, we see that the Black–Scholes–Merton equation and boundary conditions in Theorem 7.4.1 uniquely determine the function $v(t,x,y)$.