Stochastic Calculus For Finance - volume 2

- Section 5.5.1 Continuously Paying Dividend
- Section 5.5.2 Continuously Paying Dividend with Constant Coefficients
- Section 5.5.3 Lump Payments of Dividends
- Section 5.5.4 Lump Payments of Dividends with Constant Coefficients
- Section 5.6.1 Forward Contract

07/08/2025,陳亭秀

Section 5.5.1 Continuously Paying Dividend

- A stock, modeled as a generalized geometric Brownian motion
- Stock pays dividends continuously over time at a rate $A(t)$ per unit time
- $A(t), 0 \le t \le T$, is a nonnegative adapted process

Dividends paid by a stock reduce its value, and so we shall take as our model of the stock price

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt$$

An agent who holds the stock receives both the capital gain or loss due to stock price movements and the continuously paying dividend. Thus, if $\Delta(t)$ is the number of shares held at time t, then the portfolio value $X(t)$ satisfies

$$dX(t) = \Delta(t)dS(t) + \Delta(t)A(t)S(t)dt + R(t)[X(t) - \Delta(t)S(t)]dt$$

$$= R(t)X(t)dt + \big(\alpha(t) - R(t)\big)\Delta(t)S(t)dt + \sigma(t)\Delta(t)S(t)dW(t)$$

$$= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)[\Theta(t)dt + dW(t)],$$

The mean rate of return $\alpha(t)$, the volatility $\sigma(t)$, and the interest rate $R(t)$ are assumed to be adapted process

where $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$ is the usual market price of risk

We define $\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$, and use Girsanov's Theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion

$$dX(t) = R(t)X(t)dt + \Delta(t)S(t)\sigma(t)d\widetilde{W}(t)$$

The discounted portfolio value $D(t)X(t)$ satisfies

$$d[D(t)X(t)] = XdD + DdX + dDdX$$

$$= X(-RDdt) + D\left(RXdt + \Delta S\sigma d\widetilde{W}\right) + \underbrace{(-RDdt)\left(RXdt + \Delta S\sigma d\widetilde{W}\right)}_{=0}$$

$$= D(t)\Delta(t)S(t)\sigma(t)d\widetilde{W}(t) \qquad \text{(no drift term)}$$

$$D(t) = e^{-\int_0^t R(u)du}$$

=> Under the risk-neutral measure $\widetilde{\mathbb{P}}$, the discounted portfolio process is a martingale.

If we now wish to hedge a short position in a derivative security paying $V(T)$ at time T, where $V(T)$ is an $\mathcal{F}(T)$-measurable random variable, we will need to choose the initial capital $X(0)$ and the portfolio process $\Delta(t)$, $0 \le t \le T$, so that $X(T) = V(T)$. Because $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have

$$D(t)X(t) = \widetilde{\mathbb{E}}[\, D(T)V(T) \mid \mathcal{F}(t)\,], 0 \le t \le T$$

The value $X(t)$ of this portfolio at each time t is the value (price) of the derivative security at that time, which we denote by $V(t)$. Making this replacement in the formula above, we obtain the risk-neutral pricing formula

$$D(t)V(t) = \widetilde{\mathbb{E}}[\, D(T)V(T) \mid \mathcal{F}(t)\,], 0 \le t \le T$$

Same as risk-neutral pricing formula (5.2.30) in the case of no dividends. Furthermore, conditions that guarantee that a short position can be hedged, and hence risk-neutral pricing is fully justified, are the same as in the no-dividend case. (Section 5.3).

$$dS(t) = [R(t) - A(t)]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt$$
$$= [\alpha(t) - A(t)]S(t)dt + \sigma(t)S(t)dW(t)$$

By Girsanov's thm: $d\widetilde{W} = dW + \Theta(t)dt$

$$= [\alpha(t) - A(t)]S(t)dt + \sigma(t)S(t)[d\widetilde{W} - \Theta(t)dt]$$
$$= [\alpha(t) - A(t) - \sigma(t)\Theta(t)]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$
$$= \left[\alpha(t) - A(t) - \sigma(t)\frac{\alpha(t) - R(t)}{\sigma(t)}\right]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$
$$= [R(t) - A(t)]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

Under the risk-neutral measure, the stock does not have mean rate of return $R(t)$, and consequently the discounted stock price is not a martingale.

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(u)d\widetilde{W}(u) + \int_0^t \left[R(t) - A(t) - \frac{1}{2}\sigma^2(u)\right]du\right\}$$

$$D(t) = e^{-\int_0^t R(u)du}$$

$$e^{\int_0^t A(u)du}D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(u)d\widetilde{W}(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\}$$

This is a martingale. This is the interest-rate-discounted value at time *t* of an account that initially purchases one share of the stock and continuously reinvests the dividends in the stock.

Section 5.5.2 Continuously Paying Dividend with Constant Coefficients

In the event that the volatility σ, the interest rate r, and the dividend rate a are constant, the stock price at time t is

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t\right\}$$

For $0 \le t \le T$,we have

$$S(\mathrm{T}) = S(0)\exp\left\{\sigma\widetilde{W}(T) + \left(r - a - \frac{1}{2}\sigma^2\right)T\right\}$$

$$\frac{S(\mathrm{T})}{S(t)} = \exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t)\right) + \left(r - a - \frac{1}{2}\sigma^2\right)(T - t)\right\}$$

$$S(T) = S(t)\exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t)\right) + \left(r - a - \frac{1}{2}\sigma^2\right)(T - t)\right\}$$

According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T with strike K is

$$V(t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)]$$

Let $x = S(t), \tau = T - t$

$$c(t,x) = \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t)\right) + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right]$$

$$= \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left\{\sigma Y\sqrt{\tau} + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right],$$

$$Y = \frac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{T - t}}$$

Y is a standard normal random variable under $\widetilde{\mathbb{P}}$.

We define

$$d_{\pm}(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - a \pm \frac{1}{2}\sigma^2\right)\tau\right]$$

Then a European call is

$$c(t,x) = e^{-a\tau} x\, N\left(d_+(\tau,x)\right) - Ke^{-r\tau}N(d_-(\tau,x))$$

Section 5.5.3 Lump Payments of Dividends

Consider the case when the dividend is paid in lumps

- There are times $0 < t_1 < t_2 < t_n < T$ and, at each time t_j, the dividend paid is $a_j S(t_j-)$, where $S(t_j-)$ denotes the stock price just prior to the dividend payment.
- The stock price after the dividend payment is the stock price before the dividend payment less the dividend payment ;

$$S(t_j) = S(t_j -) - a_j S(t_j-) = (1 - a_j) S(t_j-)$$

We assume that each a_j is an $F(t_j) - measurable$ random variable taking values in [0,1]. If $a_j = 0$, no dividend is paid at time t_j. If a_j =1, the full value of the stock is paid as a dividend at time t_j and the stock value is zero thereafter. To simplify the notation, we set $t_0 = 0$ and $t_{n+1} = T$. However, neither $t_0 = 0$ nor $t_{n+1} = T$ is a dividend payment date (i.e., $a_0 = 0$ and $a_{n+1} = 0$). We assume that, between dividend payment dates, the stock price follows a generalized geometric Brownian motion:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), t_j \leq t < t_{j+1}, j = 0,1, \dots n$$

Between dividend payment dates, the differential of the portfolio value corresponding to a portfolio process $\Delta(t)$, $0 < t < T$, is

$$dX(t) = \Delta(t)dS(t) + R(t)[X(t) - \Delta(t)S(t)]dt$$
$$= R(t)X(t)dt + \big(\alpha(t) - R(t)\big)\Delta(t)S(t)dt + \sigma(t)\Delta(t)S(t)dW(t)$$
$$= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)[\Theta(t)dt + dW(t)],$$

where $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$ is the usual market price of risk

At the dividend payment dates, the value of the portfolio stock holdings drops by $a_j\Delta\big(t_j\big)S(t_j-)$, but the portfolio collects the dividend $a_j\Delta\big(t_j\big)S(t_j-)$, and so the portfolio value does not jump. It follows that

$$dX(t) = R(t)X(t)dt + \Delta(t)S(t)\sigma(t)[\Theta(t)dt + dW(t)]$$

is the correct formula for the evolution of the portfolio value at all times t.

Section 5.5.4 Lump Payments of Dividends with Constant Coefficients

We price a European call under the assumption that σ, r, and each a_j are constant.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

By Girsanov's thm : $d\widetilde{W} = dW + \theta(t)dt$

$$= \alpha S(t)dt + \sigma S(t)\left[d\widetilde{W}(t) - \theta(t)dt\right]$$

$$= \left[\alpha S(t) - \sigma S(t)\theta(t)\right]dt + \sigma S(t)d\widetilde{W}(t)$$

$$= \left[\alpha S(t) - \sigma S(t)\frac{\alpha - r}{\sigma}\right]dt + \sigma S(t)d\widetilde{W}(t)$$

$$= rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

$$dS(t) = rS(t)dt + \sigma(t)S(t)d\widetilde{W}(t), t_j \le t < t_{j+1}, j = 0,1,\dots,n$$

Therefore,

$$S(t_{j+1}-) = S(t_j)\exp\left\{\sigma\left(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\right) + \left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}$$

From $S(t_{j+1}) = (1 - a_{j+1})S(t_{j+1}-)$, we can see that

$$S(t_{j+1}) = (1 - a_{j+1})S(t_j)\exp\left\{\sigma\left(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\right) + \left(r - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}$$

$$\frac{S(t_{j+1})}{S(t_j)} = (1-a_{j+1})\ \exp\left\{\sigma\left(\widetilde{W}(t_{j+1})-\widetilde{W}(t_j)\right)+\left(r-\frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)\right\}$$

It follows that

$$\frac{S(T)}{S(0)} = \frac{S(t_{n+1})}{S(0)} = \prod_{j=0}^{n}\frac{S(t_{j+1})}{S(t_j)}$$

$$= \prod_{j=0}^{n}\exp\left\{\sigma\left(\widetilde{W}(t_{j+1})-\widetilde{W}(t_j)\right)+\left(r-\frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)\right\}$$

$$= \exp\left\{\sigma\left(\widetilde{W}(t_1)-\widetilde{W}(t_0)\right)+\left(r-\frac{1}{2}\sigma^2\right)(t_1-t_0)+\sigma\left(\widetilde{W}(t_2)-\widetilde{W}(t_1)\right)\right.$$
$$\left.+\left(r-\frac{1}{2}\sigma^2\right)(t_2-t_1)+\cdots+\sigma\left(\widetilde{W}(T)-\widetilde{W}(t_n)\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t_n)\right\}$$

$$= \prod_{j=0}^{n}(1-a_{j+1})\ \exp\left\{\sigma\left(\widetilde{W}(t_{j+1})-\widetilde{W}(t_j)\right)+\left(r-\frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)\right\}$$

$$S(T) = S(0)\prod_{j=0}^{n}(1-a_{j+1})\exp\left\{\sigma\widetilde{W}(T)+\left(r-\frac{1}{2}\sigma^2\right)T\right\}$$

This is the same formula we would have for the price at time T of a geometric Brownian motion not paying dividends if the initial stock price were $S(0)\prod_{j=0}^{n}(1-a_{j+1})$ rather than $S(0)$.

Therefore, the price at time zero of a European call on this dividend paying asset, a call that expires at time T with strike price K. The call price is

$$S(0)\prod_{j=0}^{n}\left(1-a_{j+1}\right)N(d_+)-Ke^{-r\tau}N(d_-)$$

$$d_{\pm}=\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K}+\sum_{j=0}^{n-1}\log(1-a_{j+1})+\left(r\pm\frac{1}{2}\sigma^2\right)T\right]$$

Section 5.6.1 Forward Contract

In this section, we assume there is a unique risk-neutral measure $\widetilde{\mathbb{P}}$, and all assets satisfy the risk-neutral pricing formula.

- Let $S(t)$, $0 \leq t \leq \bar{T}$, be an asset price process
- Let $R(t)$, $0 \leq t \leq \bar{T}$, be an interest rate process.
- Choose here some large time $\bar{T}$, and all bonds and derivative securities we consider will mature or expire at or before time $\bar{T}$.
- Define the discount process $D(t) = e^{-\int_0^t R(u)du}$

According to the risk-neutral pricing formula, the price at time t of a zero-coupon bond paying 1 at time T is $B(t,T) = \frac{1}{D(t)} \widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)], 0 \leq t \leq T \leq \bar{T}$

This pricing formula guarantees that no arbitrage can be found by trading in these bonds because any such portfolio, when discounted, will be a martingale under the risk-neutral measure.

Definition 5.6.1. A forward contract is an agreement to pay a specified delivery price K at a delivery date T, where $0 \leq t \leq T$, for the asset whose price at time *t* is *S(t)*. The *T*-forward price $For_S(t,T)$ of this asset at time *t*, where $0 \leq t \leq T \leq \bar{T}$, is the value of *K* that makes the forward contract have no-arbitrage price zero at time *t*.

Theorem5.6.2. Assume that zero-coupon bonds of all maturities can be traded. Then

$$For_S(t,T) = \frac{S(t)}{B(t,T)}, 0 \le t \le \bar{T} \quad (5.6.2)$$

Proof. Suppose that at time *t* an agent sells the forward contract with delivery date *T* and delivery price *K*. Suppose further that the value *K* is chosen so that the forward contract has price zero at time *t*. Then selling the forward contract generates no income. Having sold the forward contract at time *t*, suppose the agent immediately shorts $\frac{S(t)}{B(t,T)}$ zero-coupon bonds and uses the income $S(t)$ generated to buy one share of the asset. The agent then does no further trading until time *T*, at which time she owns one share of the asset, which she delivers according to the forward contract. In exchange, she receives K. After covering the short bond position, she is left with $K - \frac{S(t)}{B(t,T)}$. If this is positive, the agent has found an arbitrage. If it is negative, the agent could instead have taken the opposite position, going long the forward, long the *T*-maturity bond, and short the asset, to again achieve an arbitrage. In order to preclude arbitrage, *K* must be given by (5.6.2).

Remark 5.6.3. The proof of Theorem 5.6.2 does not use the notion of risk-neutral pricing. It shows that the forward price must be given by (5.6.2) in order to preclude arbitrage. Because we have assumed the existence of a risk-neutral measure and are pricing all assets by the risk-neutral pricing formula, we must be able to obtain (5.6.2) from the risk-neutral pricing formula as well. Indeed, using (5.2.30), (5.6.1), and the fact that the discounted asset price is a martingale under $\widetilde{\mathbb{P}}$, we compute the price at time *t* of the forward contract to be

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)(S(T)-K)|\mathcal{F}(t)]$$

$$=\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)S(T)|\mathcal{F}(t)]-\frac{K}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

$$=S(t)-KB(t,T)$$

In order for this to be zero, *K* must be given by (5.6.2).