Stochastic Calculus For Finance - volume 2

- Section 4.7.4 Multidimensional Distribution of the Brownian Bridge
- Section 4.7.5 Brownian Bridge as a Conditioned Brownian Motion

05/06/2025, 陳亭秀

Section 4.7.4 Multidimensional Distribution of the Brownian Bridge

We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a\to b}(t)$ denote the Brownian bridge from a to b on $[0,T]$. We also fix $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$. In this section, we compute the joint density of $X^{a\to b}(t_1)$,… , $X^{a\to b}(t_n)$.

We recall that the Brownian bridge from a to b has

the mean function: $m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T}$

and covariance function: $c(s,t) = s \wedge t - \frac{st}{T}$.

To simplify notation, we set $\tau_j = T - t_j$ so that $\tau_0 = T$. We define random variables

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}$$

Because $X^{a\to b}(t_1)$,… , $X^{a\to b}(t_n)$ are jointly normal, so are $Z(t_1), \ldots, Z(t_n)$.
We compute

1. $\mathbb{E}Z_j$
2. $Var(Z_j)$
3. $cov(Z_i, Z_j)$

to get random variable $Z(t_1), \ldots, Z(t_n)'$s joint density.

1.

$$\mathbb{E}Z_j = \mathbb{E}\left[\frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}\right]$$

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T} = \frac{\tau a}{T} + \frac{bt}{T}$$

$$= \frac{\mathbb{E}[X^{a\to b}(t_j)]}{\tau_j} - \frac{\mathbb{E}[X^{a\to b}(t_{j-1})]}{\tau_{j-1}}$$

$$= \frac{\frac{\tau_j a}{T} + \frac{bt_j}{T}}{\tau_j} - \frac{\frac{\tau_{j-1} a}{T} + \frac{bt_{j-1}}{T}}{\tau_{j-1}}$$

$$= \frac{a}{T} + \frac{bt_j}{T\tau_j} - \frac{a}{T} - \frac{bt_{j-1}}{T\tau_{j-1}}$$

$$= \frac{bt_j(T - t_{j-1}) - bt_{j-1}(T - t_j)}{T\tau_j\tau_{j-1}} = \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j-1}}$$

$$c(s,t) = s \wedge t - \frac{st}{T}$$

2. $$Var(Z_j) = Var\left[\frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}\right]$$

$$= \frac{1}{{\tau_j}^2} Var\left(X^{a\to b}(t_j)\right) - \frac{2}{\tau_j \tau_{j-1}} Cov\left(X^{a\to b}(t_j), X^{a\to b}(t_{j-1})\right) + \frac{1}{{\tau_{j-1}}^2} Var\left(X^{a\to b}(t_{j-1})\right)$$

$$= \frac{1}{{\tau_j}^2} c(t_j, t_j) - \frac{2}{\tau_j \tau_{j-1}} c(t_j, t_{j-1}) + \frac{1}{{\tau_{j-1}}^2} c(t_{j-1}, t_{j-1})$$

$$= \frac{1}{{\tau_j}^2}\left(t_j - \frac{{t_j}^2}{T}\right) - \frac{2}{\tau_j \tau_{j-1}}\left(t_{j-1} - \frac{t_j t_{j-1}}{T}\right) + \frac{1}{{\tau_{j-1}}^2}\left(t_{j-1} - \frac{{t_{j-1}}^2}{T}\right)$$

$$= \frac{1}{{\tau_j}^2}\frac{t_j \tau_j}{T} - \frac{2}{\tau_j \tau_{j-1}}\frac{t_{j-1}\tau_j}{T} + \frac{1}{{\tau_{j-1}}^2}\frac{t_{j-1}\tau_{j-1}}{T}$$

$$\frac{t_j T}{T} - \frac{{t_j}^2}{T} = \frac{t_j(T - t_j)}{T} = \frac{t_j \tau_j}{T}$$

$$= \frac{t_j}{T\tau_j} - \frac{2t_{j-1}}{T\tau_{j-1}} + \frac{t_{j-1}}{T\tau_{j-1}}$$

$$\frac{t_{j-1} T}{T} - \frac{t_j t_{j-1}}{T} = \frac{t_{j-1}(T - t_j)}{T} = \frac{t_{j-1}\tau_j}{T}$$

$$= \frac{t_j(T - t_{j-1}) - t_{j-1}(T - t_j)}{T\tau_j \tau_{j-1}}$$

$$\frac{t_{j-1} T}{T} - \frac{{t_{j-1}}^2}{T} = \frac{t_{j-1}(T - t_{j-1})}{T} = \frac{t_{j-1}\tau_{j-1}}{T}$$

$$= \frac{(t_j - t_{j-1})}{\tau_j \tau_{j-1}}$$

$$c(s,t) = s \wedge t - \frac{st}{T}$$

3. When $i < j$

$$Cov(Z_i, Z_j) = Cov\left(\frac{X^{a\to b}(t_i)}{\tau_i} - \frac{X^{a\to b}(t_{i-1})}{\tau_{i-1}}, \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}\right)$$

$$= \frac{1}{\tau_i\tau_j} Cov\left(X^{a\to b}(t_i), X^{a\to b}(t_j)\right) - \frac{1}{\tau_i\tau_{j-1}} Cov\left(X^{a\to b}(t_i), X^{a\to b}(t_{j-1})\right)$$

$$- \frac{1}{\tau_{i-1}\tau_j} Cov\left(X^{a\to b}(t_{i-1}), X^{a\to b}(t_j)\right) + \frac{1}{\tau_{i-1}\tau_{j-1}} Cov\left(X^{a\to b}(t_{i-1}), X^{a\to b}(t_{j-1})\right)$$

$$= \frac{1}{\tau_i\tau_j} c(t_i, t_j) - \frac{1}{\tau_i\tau_{j-1}} c(t_i, t_{j-1}) - \frac{1}{\tau_{i-1}\tau_j} c(t_{i-1}, t_j) + \frac{1}{\tau_{i-1}\tau_{j-1}} c(t_{i-1}, t_{j-1})$$

$$= \frac{1}{\tau_i\tau_j}\left(t_i - \frac{t_i t_j}{T}\right) - \frac{1}{\tau_i\tau_{j-1}}\left(t_i - \frac{t_i t_{j-1}}{T}\right) - \frac{1}{\tau_{i-1}\tau_j}\left(t_{i-1} - \frac{t_{i-1} t_j}{T}\right) + \frac{1}{\tau_{i-1}\tau_{j-1}}\left(t_{i-1} - \frac{t_{i-1} t_{j-1}}{T}\right)$$

$$= \frac{t_i(T - t_j)}{T\tau_i\tau_j} - \frac{t_i(T - t_{j-1})}{T\tau_i\tau_{j-1}} - \frac{t_{i-1}(T - t_j)}{T\tau_{i-1}\tau_j} + \frac{t_{i-1}(T - t_{j-1})}{T\tau_{i-1}\tau_{j-1}}$$

$$= 0$$

$\begin{cases}\text{Joint Gaussian} \\ \text{Uncorrelated}\end{cases} \Rightarrow$ independent

⇨ We conclude that the normal random variables $Z_1, \dots, Z_n$ are independent

Because $X^{a\to b}(t_1)$,… , $X^{a\to b}(t_n)$ are jointly normal, so are $Z(t_1), \dots, Z(t_n)$.And, $Z_1, \dots, Z_{\mathrm{n}}$ are independent. $Z_1, \dots, Z_{\mathrm{n}}$'s joint density, which is

$$f_{Z(t_1),\dots,Z(t_n)}(z_1,\dots,z_n) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{(t_j - t_{j-1})}{\tau_j \tau_{j-1}}}} exp\left\{-\frac{1}{2}\frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}}\right)^2}{\frac{(t_j - t_{j-1})}{\tau_j \tau_{j-1}}}\right\}$$

$$= exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}}\right)^2}{\frac{(t_j - t_{j-1})}{\tau_j \tau_{j-1}}}\right\} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{(t_j - t_{j-1})}{\tau_j \tau_{j-1}}}}$$

Make the change of variables $z_j = \frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}$

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}$$

$$\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}}\right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}}$$

$$= \sum_{j=1}^{n} \frac{\tau_j \tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}}\right)^2$$

$$= \sum_{j=1}^{n} \frac{\tau_j \tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j^2}{\tau_j^2} + \frac{x_{j-1}^2}{\tau_{j-1}^2} + \frac{b^2(t_j - t_{j-1})^2}{\tau_j^2 \tau_{j-1}^2} - \frac{2x_j x_{j-1}}{\tau_j \tau_{j-1}} - \frac{2x_j b(t_j - t_{j-1})}{\tau_j^2 \tau_{j-1}} + \frac{2x_{j-1} b(t_j - t_{j-1})}{\tau_j \tau_{j-1}^2}\right)$$

$$= \sum_{j=1}^{n} \left(\frac{\tau_{j-1} x_j^2}{\tau_j (t_j - t_{j-1})} + \frac{\tau_j x_{j-1}^2}{\tau_{j-1}(t_j - t_{j-1})} + \frac{b^2(t_j - t_{j-1})}{\tau_j \tau_{j-1}} - \frac{2x_j x_{j-1}}{t_j - t_{j-1}} - \frac{2x_j b}{\tau_j} + \frac{2x_{j-1} b}{\tau_{j-1}}\right)$$

$$= \sum_{j=1}^{n} \left[\frac{x_j^2}{t_j - t_{j-1}} \left(1 + \frac{\tau_{j-1} - \tau_j}{\tau_j}\right) + \frac{x_{j-1}^2}{t_j - t_{j-1}} \left(1 - \frac{\tau_{j-1} - \tau_j}{\tau_{j-1}}\right) - \frac{2x_j x_{j-1}}{t_j - t_{j-1}}\right]$$

$$+ b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}}\right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}\right).$$

$$\tau_{j-1} - \tau_j = (T - t_{j-1}) - (T - t_j) = t_j - t_{j-1},$$

$$\sum_{j=1}^{n}\left[\frac{x_j^2-2x_jx_{j-1}+x_{j-1}^2}{t_j-t_{j-1}}\right]+\sum_{j=1}^{n}\left(\frac{x_j^2}{\tau_j}-\frac{x_{j-1}^2}{\tau_{j-1}}\right)+b^2\sum_{j=1}^{n}\left(\frac{1}{\tau_j}-\frac{1}{\tau_{j-1}}\right)-2b\sum_{j=1}^{n}\left(\frac{x_j}{\tau_j}-\frac{x_{j-1}}{\tau_{j-1}}\right)$$

$$=\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+\frac{x_n^2}{T-t_n}-\frac{a^2}{T}+b^2\left(\frac{1}{T-t_n}-\frac{1}{T}\right)-2b\left(\frac{x_n}{T-t_n}-\frac{a}{T}\right)$$

$$=\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+\frac{(b-x_n)^2}{T-t_n}-\frac{(b-a)^2}{T}.$$

$$exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{\left(z_j-\frac{b(t_j-t_{j-1})}{\tau_j\tau_{j-1}}\right)^2}{\frac{(t_j-t_{j-1})}{\tau_j\tau_{j-1}}}\right\}=\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+\frac{(b-x_n)^2}{T-t_n}-\frac{(b-a)^2}{T}\right\}$$

To change a density, we also need to account for the Jacobian of the change of variables. In this case, we have

$$z_j = \frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}$$

$$\frac{\partial z_j}{\partial x_j} = \frac{1}{\tau_j}, \quad j = 1, \dots, n,$$

$$\frac{\partial z_j}{\partial x_{j-1}} = -\frac{1}{\tau_{j-1}}, \quad j = 2, \dots, n,$$

and all other partial derivatives are zero. This leads to the Jacobian matrix

$$J = \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 & \dots & 0 \\ -\frac{1}{\tau_1} & \frac{1}{\tau_2} & 0 & \dots & 0 \\ 0 & -\frac{1}{\tau_2} & \frac{1}{\tau_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\tau_n} \end{bmatrix}$$

whose determinant is $\prod_{j=1}^{n} \frac{1}{\tau_j}$.

We obtain the density for $X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)$

$$\prod_{j=1}^{n}\frac{1}{\tau_j}\cdot\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi\frac{(t_j-t_{j-1})}{\tau_j\tau_{j-1}}}}$$

$$f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1,\ldots,x_n)$$

$$=\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\sqrt{\frac{\tau_{j-1}}{\tau_j}}exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+\frac{(b-x_n)^2}{T-t_n}-\frac{(b-a)^2}{T}\right\}$$

$$=\sqrt{\frac{T}{T-t_n}}\cdot\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+\frac{(b-x_n)^2}{T-t_n}-\frac{(b-a)^2}{T}\right\}$$

$$=\frac{p(T-t_n,x_n,b)}{p(T,a,b)}\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j)$$

where

$$p(\tau,x,y)=\frac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{(y-x)^2}{2\tau}\right\}$$

is the transition density for Brownian Motion

Transition density：在時間段 τ ，從x到y的機率

Section 4.7.5 Brownian Bridge as a Conditioned Brownian Motion

The joint density (4.7.6) for $X^{a\to b}(t_1), \dots, X^{a\to b}(t_n)$ permits us to give one more interpretation for the Brownian bridge from a to b on $[0, T]$. It is a Brownian motion $W(t)$ on this time interval, starting at $W(0) = a$ and conditioned to arrive at b at time T (i.e., conditioned on $W(T) = b$). Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$ be given. The joint density of $W(t_1), \dots, W(t_n), W(T)$ is

$$f_{W(t_1),\dots,W(t_n),W(T)}(x_1, \dots, x_n, b) = p(T - t_n, x_n, b) \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j), \quad (4.7.7)$$

where $W(0) = x_0 = a$. This is because $p(t_1 - t_0, x_0, x_1) = p(t_1, a, x_1)$ is the density for the Brownian motion going from $W(0) = a$ to $W(t_1) = x_1$ in the time between $t = 0$ and $t = t_1$. Similarly, $p(t_2 - t_1, x_1, x_2)$ is the density for going from $W(t_1) = x_1$ to $W(t_2) = x_2$ between time $t = t_1$ and $t = t_2$. The joint density for $W(t_1)$ and $W(t_2)$ is then the product

$$p(t_1, a, x_1)p(t_2 - t_1, x_1, x_2).$$

Continuing in this way, we obtain the joint density (4.7.7). The marginal density of $W(T)$ is $p(T,a,b)$. The density of $W(t_1),\ldots,W(t_n)$ conditioned on $W(T)=b$ is thus the quotient

$$\frac{p(T-t_n,x_n,b)}{p(T,a,b)}\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j),$$

Brownian Bridge可以看作是一種 Conditional Brownian Motion。

and this is $f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1,\ldots,x_n)$ of (4.7.6).

Finally, let us define

$$M^{a\to b}(T)=\max_{0\le t\le T}X^{a\to b}(t)$$

to be the maximum value obtained by the Brownian bridge from a to b on $[0,T]$. This random variable has the following distribution.

Corollary 4.7.7. *The density of $M^{a \to b}(T)$ is*

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y-b-a)}{T} e^{-\frac{2}{T}(y-a)(y-b)}, \qquad y > \max\{a, b\}. \tag{4.7.8}$$

Proof. Because the Brownian bridge from 0 to w on $[0, T]$ is a Brownian motion conditioned on $W(T) = w$, the maximum of $X^{0\to w}$ on $[0, T]$ is the maximum of W on $[0, T]$ conditioned on $W(T) = w$. Therefore, the density of $M^{0\to w}(T)$ was computed in Corollary 3.7.4 and is

Corollary 3.7.4. *The conditional distribution of $M(t)$ given $W(t) = w$ is* $M(t) = \max_{0\le s\le t} W(s)$.

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}, \qquad w \le m, m > 0.$$

$$f_{M^{0\to w}(T)}(m) = \frac{2(2m-w)}{T} e^{-\frac{2m(m-w)}{T}}, \qquad w < m, m > 0. \tag{4.7.9}$$

The density of $f_{M^{a\to b}(T)}(y)$ can be obtained by translating from the initial condition $W(0) = a$ to $W(0) = 0$ and using (4.7.9). In particular, in (4.7.9) we replace m by $y - a$ and replace w by $b - a$. This results in (4.7.8). □