Stochastic Calculus For Finance - volume 2

- Section 4.3 Ito’s Integral for General Integrands
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Section 4.3 Ito's Integral for General Integrands

Define the Ito integral $\int_0^T \Delta(t)\, dW(t)$ for integrands $\Delta(t)$ that are allowed to vary continuously with time and also to jump.

We do assume that $\Delta(t)$, t ≥ 0, is adapted to the filtration $\mathcal{F}$(t), t ≥ 0. We also assume the square-integrability condition $E[\int_0^T \Delta^2(t)dt] < \infty$.

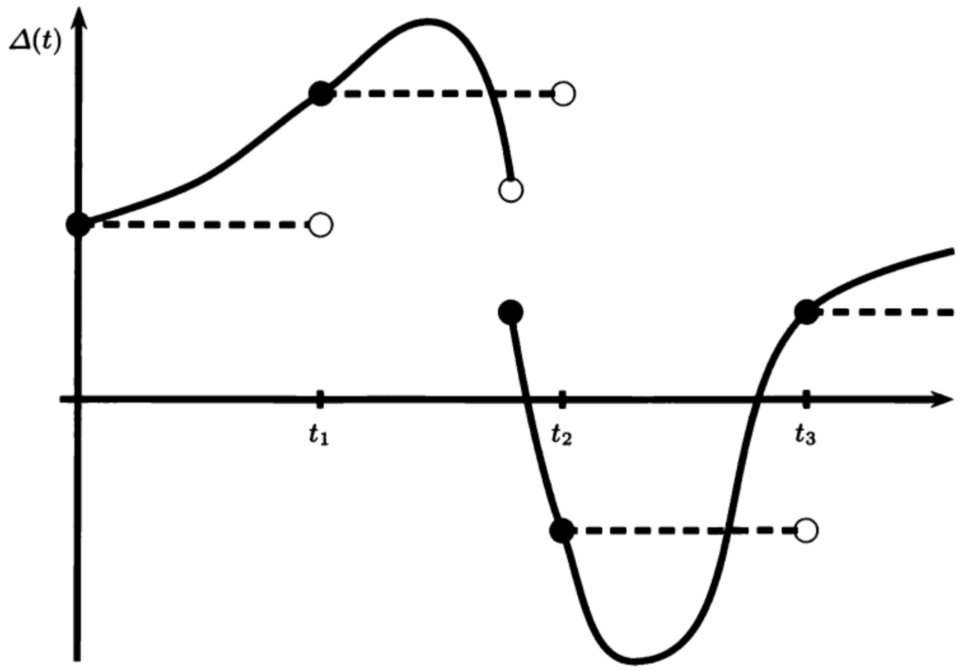

Fig. 4.3.1. Approximating a continuously varying integrand.

Constructed by choosing a partition $0 = t_0 < t_1 < t_2 < t_3 < t_4$, setting the approximating simple process equal to $\Delta(t_j)$ at each t_j , and then holding the simple process constant over the subinterval $[t_j , t_{j+1})$.
As the maximal step size of the partition approaches zero , the approximating integrand will become a better and better approximation of the continuously varying one.

In general, then, it is possible to choose a sequence $\Delta_n(t)$ of simple processes such that as $n \to \infty$ these processes converge to the continuously varying $\Delta(t)$.

$$\lim_{n\to\infty} E\left[\int_0^T |\Delta_n(t) - \Delta(t)|^2 \, dt\right] = 0$$

Define the Ito integral for the continuously varying integrand $\Delta(t)$ by the formula

$$\int_0^T \Delta(u) dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(t) \, dW(u) , 0 \le t \le T \qquad (4.3.3)$$

Theorem 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies (4.3.1). Then $I(t) = \int_0^t \Delta(u)dW(u)$ defined by (4.3.3) has the following properties.*

$\mathrm{E}[\int_0^T \Delta^2(t)dt] < \infty$

(i) **(Continuity)** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*

(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)dW(u)$.*

(iv) **(Martingale)** *$I(t)$ is a martingale,*

(v) **(Itô isometry)** $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$.

(vi) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u)du$.

Example 4.3.2 Compute $\int_0^t W(t)dW(t)$

To do that, we choose a large integer n and approximate the integrand $\Delta(t) = W(t)$ by the simple process

$$\Delta_n(t) = \begin{cases} W(0) = 0 & \text{if } 0 \le t < \frac{T}{n}, \\ W(\frac{T}{n}) & \text{if } \frac{T}{n} \le t < \frac{2T}{n}, \\ \vdots & \\ W\left(\frac{(n-1)T}{n}\right) & \text{if } \frac{(n-1)T}{n} \le t < T, \end{cases}$$

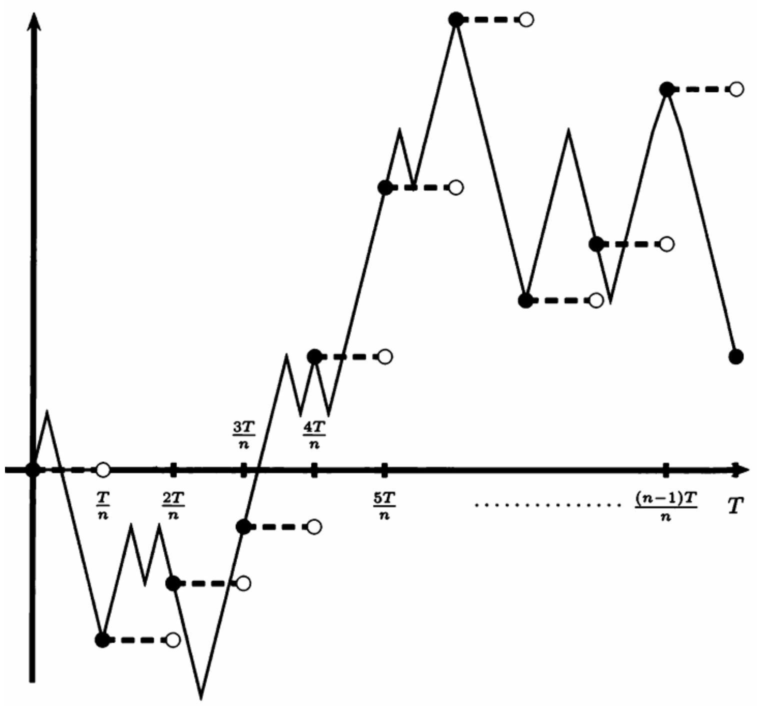

Fig. 4.3.2. Simple process approximating Brownian motion.

Then $\lim_{n\to\infty} E \int_0^T |\Delta_n(t) - W(t)|^2 dt = 0$. By definition,

$$\begin{aligned} \int_0^t W(t)dW(t) &= \lim_{n\to\infty} \int_0^t \Delta_n(t)dW(t) \\ &= \lim_{n\to\infty} \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]. \end{aligned} \quad (4.3.4)$$

To simplify notation, we denote $W_j = W(\frac{jT}{n})$, $W(0) = 0$.

$$\frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2 = \frac{1}{2}\sum_{j=0}^{n-1}W_{j+1}^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2$$

$$= \frac{1}{2}\sum_{k=1}^{n}W_k^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2$$

$$= \frac{1}{2}W_n^2 + \frac{1}{2}\sum_{k=0}^{n-1}W_k^2 - \sum_{j=0}^{n-1}W_jW_{j+1} + \frac{1}{2}\sum_{j=0}^{n-1}W_j^2 \qquad (4.3.5)$$

$$= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1}W_j^2 - \sum_{j=0}^{n-1}W_jW_{j+1}$$

$$= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1}W_j(W_j - W_{j+1}).$$

$$\Rightarrow \quad \sum_{j=0}^{n-1}W_j(W_{j+1}-W_j) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2.$$

$$\int_0^t W(t)dW(t) = \lim_{n\to\infty}\int_0^t \Delta_n(t)dW(t)$$

$$= \lim_{n\to\infty}\sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]. \qquad (4.3.4)$$

$$= \frac{1}{2}W^2(T) - \frac{1}{2}\sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2.$$

$$= \frac{1}{2}W^2(T) - \frac{1}{2}[W,W](T) = \frac{1}{2}W^2(T) - \frac{1}{2}T. \qquad (4.3.6)$$

We contrast (4.3.6) with ordinary calculus. If g is a differentiable function with g(0)=0, then

$$\int_0^T g(t)dg(t) = \int_0^T g(t)g'(t)dt = \frac{1}{2}g^2(t)\Big|_0^T = \frac{1}{2}g^2(T).$$

The extra term $-\frac{1}{2}T$ in (4.3.6) comes from the nonzero quadratic variation of Brownian motion and the way we constructed the Ito integral, always evaluating the integrand at the left-hand endpoint of the subinterval (see the right-hand side of (4.3.4)). If we were instead to evaluate at the midpoint, replacing the right-hand side of (4.3.4) by

$$\lim_{n\to\infty}\sum_{j=0}^{n-1} W\left(\frac{(j+\frac{1}{2})T}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right], \qquad (4.3.7)$$

then we would not have gotten this term (see Exercise 4.4). The integral obtained by making this replacement is called the *Stratonovich integral*, and the ordinary rules of calculus apply to it. However, it is inappropriate for finance.

The upper limit of integration T in (4.3.6) is arbitrary and can be replaced by any $t \geq 0$. In other words,

$$\int_0^t W(u)dW(u) = \frac{1}{2}W^2(t) - \frac{1}{2}t, \qquad t \geq 0. \tag{4.3.8}$$

Theorem 4.3.1(iv) guarantees that $\int_0^t W(u)dW(u)$ is a martingale and hence has constant expectation. At $t = 0$, this martingale is 0, and hence its expectation must always be zero. This is indeed the case because $\mathbb{E}W^2(t) = t$. If the term $-\frac{1}{2}t$ were not present, we would not have a martingale.

Section 4.4.1 Ito-Doeblin Formula for Brownian Motion

We want a rule to "differentiate" expressions of the form $f(W(t))$, where $f(x)$ is a differentiate function and $W(t)$ is a Brownian motion. If $W(t)$ were also differentiate, then the chain rule from ordinary calculus would give

$$\frac{d}{dt} f\big(W(t)\big) = f'\big(W(t)\big)W'(t) \quad \text{or}$$

$$df\big(W(t)\big) = f'\big(W(t)\big)W'(t)dt = f'\big(W(t)\big)dW(t)$$

W has nonzero quadratic variation, the correct formula has an extra term, it is the Ito-Doeblin formula in differential form

$$df\big(W(t)\big) = f'\big(W(t)\big)dW(t) + \frac{1}{2}f''\big(W(t)\big)dt$$

Integrating this, we obtain the Ito-Doeblin formula in integral form:

$$f\big(W(t)\big) - f\big(W(0)\big) = \underbrace{\int_0^t f'\big(W(u)\big)dW(u)}_{\text{Ito integral}} + \frac{1}{2}\underbrace{\int_0^t f''\big(W(u)\big)du}_{\text{ordinary (Lebesgue) integral with respect to the time variable}}$$

Ito integral

ordinary (Lebesgue) integral with respect to the time variable

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$f\big(T, W(T)\big) = f\big(0, W(0)\big) + \int_0^T f_t\big(t, W(t)\big)\,dt + \int_0^T f_x\big(t, W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t, W(t)\big)\,dt. \tag{4.4.3}$$

- first show (4.4.3) holds when $f(x) = \frac{1}{2}x^2$.

$f'(x) = x \quad f''(x) = 1$. Let x_{j+1} and x_j be numbers.

$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2}f''(x_j)(x_{j+1} - x_j)^2$ ... Taylor's formula (4.4.4)

We are interested in the difference between $f(W(0))$ and $f(W(T))$

Fix $T > 0$, let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$

We use 4.4.4 with $x_j = W(t_j) \quad x_{j+1} = W(t_{j+1})$

$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} [f(W(t_{j+1})) - f(W(t_j))]$$

$$= \sum_{j=0}^{n-1} f'(W(t_j))[W(t_{j+1}) - W(t_j)] + \frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))[W(t_{j+1}) - W(t_j)]^2 \quad (4.4.5)$$

$f(x) = \frac{1}{2}x^2$, $(4.4.5) = \sum_{j=0}^{n-1} W(t_j)[W(t_{j+1}) - W(t_j)] + \frac{1}{2}\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2$

When $\|\Pi\| \to 0$, $f(W(T)) - f(W(0)) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} W(t_j)[W(t_{j+1}) - W(t_j)] + \lim_{\|\Pi\| \to 0} \frac{1}{2}\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2$

quadratic variation of Brownian Motion

$$= \int_0^T W(t)\,dW(t) + \frac{1}{2}T$$

$$= \int_0^T f'(W(t))\,dW(t) + \frac{1}{2}\int_0^T f''(W(t))\,dt$$

- If we take a function $f(t,x)$ of both the time variable t and the variable x, then Taylor's thm says that

$$f(t_{j+1},x_{j+1})-f(t_j,x_j)=f_t(t_j,x_j)(t_{j+1}-t_j)+f_x(t_j,x_j)(x_{j+1}-x_j)$$
$$+\frac{1}{2}f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2+\frac{1}{2}f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2$$
$$+f_{tx}(t_j,x_j)(t_{j+1}-t_j)(x_{j+1}-x_j)+\cdots \qquad (4.4.8)$$

We replace x_j by $W(t_j)$, x_{j+1} by $W(t_{j+1})$

$$f(T,W(T))-f(0,W(0))=\sum_{j=0}^{n-1}\left[f(t_{j+1},W(t_{j+1}))-f(t_j,W(t_j))\right]$$

$$=\underbrace{\sum_{j=0}^{n-1}f_t(t_j,W(t_j))(t_{j+1}-t_j)}_{①}+\underbrace{\sum_{j=0}^{n-1}f_x(t_j,W(t_j))(W(t_{j+1})-W(t_j))}_{②}$$
$$+\underbrace{\frac{1}{2}\sum_{j=0}^{n-1}f_{tt}(t_j,W(t_j))(t_{j+1}-t_j)^2}_{③}+\underbrace{\frac{1}{2}\sum_{j=0}^{n-1}f_{xx}(t_j,W(t_j))(W(t_{j+1})-W(t_j))^2}_{④}$$
$$+\underbrace{\sum_{j=0}^{n-1}f_{tx}(t_j,W(t_j))(t_{j+1}-t_j)(W(t_{j+1})-W(t_j))}_{⑤}+\cdots$$

$\to t_{j+1}-t_j$

$\|\Pi\|\to 0$

take the limit as $\|\Pi\|\to 0$

① $=\int_0^T f_t(t,W(t))\,dt$ (Ordinary integral)

② $=\int_0^T f_x(t,W(t))\,dW(t)$ (Itô integral)

③ $=0$

④ $=\frac{1}{2}\int_0^T f_{xx}(t,W(t))\,dt$

⑤ $=0$

③ $$\lim_{\|\Pi\|\to 0}\left|\frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j,W(t_j))\,(t_{j+1}-t_j)^2\right| \le \lim_{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}\left|f_{tt}(t_j,W(t_j))\right|(t_{j+1}-t_j)^2$$

$$\le \frac{1}{2}\lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}(t_{k+1}-t_k)\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f_{tt}|\,(t_{j+1}-t_j)$$

$$=\frac{1}{2}\times 0\times\int_0^T|f_{tt}(t,W(t))|\,dt = 0$$

⑤ $$\lim_{\|\Pi\|\to 0}\left|\sum_{j=0}^{n-1} f_{tx}(t_j,W(t_j))(t_{j+1}-t_j)(W(t_{j+1})-W(t_j))\right| \le \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tx}(t_j,W(t_j))\right|\times(t_{j+1}-t_j)\times\left|W(t_{j+1})-W(t_j)\right|$$

$$\le \lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}(W(t_{j+1})-W(t_j))\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f_{tx}|\,(t_{j+1}-t_j)$$

$$= 0\times\int_0^T|f_{tx}(t,W(t))|\,dt = 0$$

$$f(T,W(T))-f(0,W(0)) = \int_0^T f_t\,dt+\int_0^T f_x\,dW(t)+\frac{1}{2}\int_0^T f_{xx}\,dt$$ #

Remark 4.4.2. The fact that the sum (4.4.10) of terms containing the product $(t_{j+1} - t_j)\big(W(t_{j+1}) - W(t_j)\big)$ has limit zero can be informally recorded by the formula $dt dW(t) = 0$. Similarly, the sum (4.4.11) of terms containing $(t_{j+1} - t_j)^2$ also has limit zero, and this can be recorded by the formula $dt dt = 0$. We can write these terms if we like in the Itô-Doeblin formula, so that in differential form it becomes

$$df\big(t, W(t)\big) = f_t\big(t, W(t)\big)dt + f_x\big(t, W(t)\big)dW(t) + \frac{1}{2}f_{xx}\big(t, W(t)\big)dW(t)dW(t)$$
$$+ f_{tx}\big(t, W(t)\big)dt dW(t) + \frac{1}{2}f_{tt}\big(t, W(t)\big)dt dt,$$

but

$$dW(t)dW(t) = dt, \qquad dt dW(t) = dW(t)dt = 0, \qquad dt dt = 0, \tag{4.4.12}$$

and the Itô-Doeblin formula in differential form simplifies to

$$df\big(t, W(t)\big) = f_t\big(t, W(t)\big)dt + f_x\big(t, W(t)\big)dW(t) + \frac{1}{2}f_{xx}\big(t, W(t)\big)dt. \tag{4.4.13}$$