8.5

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8.5 American Call

Treat the American call on the usual geometric Brownian motion asset of (8.3.1)

and a variation of this asset that pays dividends at discrete dates.

Subsection 8.5.1: For non-dividend assets, American and European call prices are identical.

Subsection 8.5.2: For dividend-paying assets, we provide a recursion formula for American calls.

8.5.1 Underiying Asset Pays No Dividends

Consider a stock whose price process S(t) is given by $\boldsymbol{dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)}$ **(8.5.1)**

Where r , $\sigma > 0$ and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\tilde{P}$.

Lemma 8.5.1.

Let $h(x)$ be a nonnegative,convex function of $x \geq 0$ satisfying $h(0) \; = \; 0$.

Then the discounted intrinsic value $e^{-rt}h(S(t))$ of the American derivatiue security

that pays $h(S(t))$ upon exercise is a **submartingale**.

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Proof：

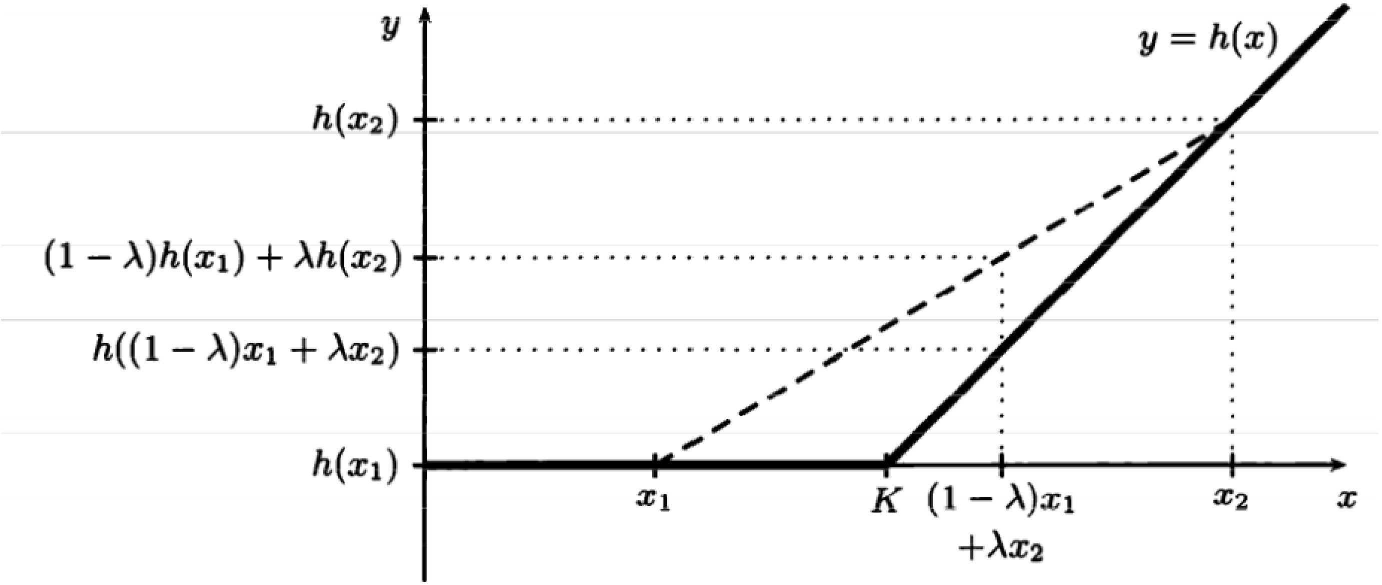

Fig. 8.5.1. The convex function $h(x) = (x - K)^+$.

Because $h(x)$ is convex,for $0 \le \lambda \le 1$ and $0 \le x_1 \le x_2$

That $\boldsymbol{h\big((1-\lambda)x_1 + \lambda x_2\big) \le (1-\lambda)h(x_1) + \lambda h(x_2)}$ **(8.5.2)**

Taking $x_1 = 0, x_2 = x$,and using the fact that h(0) = 0,*we* obtain from (8.5.2) that

$\boldsymbol{h(\lambda x) \le \lambda h(x)}$ **for all** $\boldsymbol{x \ge 0, 0 \le \lambda \le 1}$ **(8.5.3)**

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For $0 \leq u \leq t \leq T$, we have $0 \leq e^{-r(t-u)} \leq 1$, and (8.5.3) implies

$$\widetilde{\boldsymbol{E}}\,[\boldsymbol{e}^{-r(t-u)}\boldsymbol{h}(\boldsymbol{S}(\boldsymbol{t}))|\boldsymbol{F}(\boldsymbol{u})] \geq \widetilde{\boldsymbol{E}}\,[\boldsymbol{h}(\boldsymbol{e}^{-r(t-u)}\boldsymbol{S}(\boldsymbol{t}))|\boldsymbol{F}(\boldsymbol{u})] \qquad \textbf{(8.5.4)}$$

The conditional Jensen's inequality implies

$$\widetilde{\boldsymbol{E}}[\boldsymbol{h}(\boldsymbol{e}^{-r(t-u)}\boldsymbol{S}(\boldsymbol{t}))|\boldsymbol{F}(\boldsymbol{u})] \geq \mathbf{h}\left(\widetilde{\boldsymbol{E}}\left[\boldsymbol{e}^{-r(t-u)}\boldsymbol{S}(\boldsymbol{t})\middle|\boldsymbol{F}(\boldsymbol{u})\right]\right) = \boldsymbol{h}(\boldsymbol{e}^{ru}\widetilde{\boldsymbol{E}}[\boldsymbol{e}^{-rt}\boldsymbol{S}(\boldsymbol{t})|\boldsymbol{F}(\boldsymbol{u})]) \qquad \textbf{(8.5.5)}$$

Because $e^{-rt}S(t)$ is a martingale under $\widetilde{P}$, we have

$$\boldsymbol{h}(\boldsymbol{e}^{ru}\widetilde{\boldsymbol{E}}[\boldsymbol{e}^{-rt}\boldsymbol{S}(\boldsymbol{t})|\boldsymbol{F}(\boldsymbol{u})])=\boldsymbol{h}(\boldsymbol{e}^{ru}\boldsymbol{e}^{-ru}\boldsymbol{S}(\boldsymbol{u})) = \boldsymbol{h}\big(\boldsymbol{S}(\boldsymbol{u})\big) \qquad \textbf{(8.5.6)}$$

Putting (8.5.4)、(8.5.6) together, we conclude that

$\widetilde{\boldsymbol{E}}[\boldsymbol{e}^{-r(t-u)}\boldsymbol{h}(\boldsymbol{S}(\boldsymbol{t}))|\boldsymbol{F}(\boldsymbol{u})] \geq \boldsymbol{h}(\boldsymbol{S}(\boldsymbol{u}))$ **(8.5.7)** equivalently $\widetilde{\boldsymbol{E}}[\boldsymbol{e}^{-rt}\boldsymbol{h}(\boldsymbol{S}(\boldsymbol{t}))|\boldsymbol{F}(\boldsymbol{u})] \geq \boldsymbol{e}^{-ru}\boldsymbol{h}(\boldsymbol{S}(\boldsymbol{u}))$ **(8.5.8))**

This is the submartingale property $e^{-rt}h(S(t))$

Theorem 8.5.2.

Let $h(x)$ be a nonnegative, convex function of $x \geq 0$ satisfying $h(0) = 0$.

The price of an American derivative expiring at T with intrinsic value

$h(S(t)), 0 \leq t \leq T$ equals the price of the corresponding European derivative.

PROOF:

Replacing t by T in (8.5.7), we obtain => $\boldsymbol{\widetilde{E}\left[e^{-r(T-u)}h\big(S(T)\big)\middle|F(u)\right] \geq h\big(S(u)\big), 0 \leq u \leq T}$

This shows that the option to exercise early is worthless, and the price of the American derivative security agrees with the price of the European security.

Corollary 8.5.3.

The price of an American call on an asset not paying a dividend is the same as the price of the European call on the same asset with the same expirotion.

PROOF: Take $h(x) = (x - K)^+$ in Theorem 8.5.2.

Because $h(x) = (x - K)^+$ satisify Theorem 8.5.2. that American call = European call

That $e^{-rt}h(S(t)) = e^{-rt}(S(t) - K)^+$ is a submartingale under $\tilde{P}$ and hence tends to rise.

Therefore, it is optimal to wait until expiration before deciding whether to exercise.

Corollary 8.5.3.

There are two factors that contribute to the submartingale property for $e^{-rt}(S(t)-K)^{+}$.

One is $e^{-rt}\mathrm{K}$ and another one is $e^{-rt}S(t)$.

In fact, $e^{-rt}(S(t)-K)$ is a submartingale because $e^{-rt}S(t)$ is a martingale under $\tilde{P}$ and $e^{-rt}K$ increases as t increases. When we reinstate the +, we are taking a convex function of a submartingale and, because of Jensen's inequality,

$\tilde{E}[h\big(e^{-rt}(S(T)-K)\big)] \geq h(\tilde{E}[e^{-rt}(S(T)-K)\ ])$ this reinforces the upward trend.

8.5.1 Underiying Asset Pays No Dividends

The previous argument does not apply to the American put, whose discounted intrinsic value $e^{-rt}(K - S(t))$ (without the +) is a supermartingale

$e^{-rt}K$ falls and $-e^{-rt}S(t)$ is a martingale). Jensen's inequality creates an upward trend that competes with this supermartingale property, and the analysis becomes complicated.

8.5.2 Underlying Asset Pays Dividends

Consider an American call on an asset governed by (8.5.1) between dividend dates.

Assume times $0 < t_1 < t_2 < \cdots < t_n < T$, with dividends $a_j S(t_j-)$ paid at each t_j,

where $S(t_j-)$ is the pre-dividend price.

The post-dividend price is : $\boldsymbol{S(t_j) = S(t_j -) - a_j S(t_j-) = (1 - a_j) S(t_j-)}$. **(8.5.9)**

Assume $a_j \epsilon (0,1)$,set $t_0 = 0$, and that neither t_0 nor T are dividend dates.

Optimal exercise occurs only immediately before dividend payments.

- Between payments, the price satisfies the Black-Scholes-Merton PDE.
- At payment dates, the price is the maximum of the intrinsic value and the post-dividend call value.

These observations yield a recursive pricing algorithm.

8.5.2 Underlying Asset Pays Dividends

For $t_j < t < t_{j+1}$, we have $S(t) = S(t_j) \exp\{\sigma(\widetilde{W}(t) - \widetilde{W}(t_j)) + (r - \frac{1}{2}\sigma^2)(t - t_j)\}$,

which implies

$$\boldsymbol{S(t_{j+1}-) = S(t_j) exp\{\sigma(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)) + (r - \frac{1}{2}\sigma^2)(t_{j+1} - t_j)\}} \quad \textbf{(8.5.10)}$$

And

$$\boldsymbol{S(t_{j+1}) = (1 - \alpha_{j+1}) S(t_j) exp\{\sigma(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)) + (r - \frac{1}{2}\sigma^2)(t_{j+1} - t_j)\}} \quad \textbf{(8.5.11)}$$

We also have

$$\boldsymbol{S(T) = S(t_n) exp\{\sigma(\widetilde{W}(T) - \widetilde{W}(t_n)) + (r - \frac{1}{2}\sigma^2)(T - t_n)\}} \quad \textbf{(8.5.12)}$$

8.5.2 Underlying Asset Pays Dividends

Consider an American call with expiry T and strike K.

For $t_n \le t \le T$, the discounted asset $e^{-rt}S(t)$ is a martingale under $\tilde{P}$,

and Lemma 8.5.1 can be invoked to show that $e^{-rt}(S(t)-K)^+$ is a submartingale.

Therefore

$\boldsymbol{\widetilde{E}\left[e^{-rT}(S(T)-K)^+ \middle| F(t)\right] \ge e^{-rt}(S(t)-K)^+, t_n \le t \le T}$ **(8.5.13)**

This shows that, for all $t \in [t_n, T]$, the price of the European call at time t,

$c_n\,(t, S(t)) = \tilde{E}\,[e^{-r(t-t)}(S(T)-K)^+|F(t)]$

is greater than the intrinsic value of the American call, $(S(t)— K)^+$.

8.5.1 Underiying Asset Pays No Dividends

For $t_n \le t \le T$, American and European call prices at time are identical

, given by the Black-Scholes-Merton formula (8.5.14)：

$$\boldsymbol{c_n(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N(d_-(T-t,x))} \quad \textbf{(8.5.14)}$$

where $d_\pm(r,x) = \frac{1}{\sigma\sqrt{r}}[log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)r]$

Although one cannot simply substitute $x = 0$ into (8.5.14), but we have $c(t,0) = 0$

8.5.1 Underiying Asset Pays No Dividends

Formula (8.5.14) is derived from the conditional expectation in (8.5.13) by setting $S(t) = x$.

At $t = t_n$, applying (8.5.12) leads to:

$$\boldsymbol{c_n(t_n, x) = \widetilde{E}[e^{-r(T-t_n)}\left(x \exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t_n)\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t_n)\right\} - K\right)^+]} \textbf{ (8.5.15)}$$

Where $\widetilde{W}(T) - \widetilde{W}(t_n) \sim N(0, T - t_n)$ and let $\tau = T - t_n$ 、$Z = \frac{\widetilde{W}(T) - \widetilde{W}(t_n)}{\sqrt{\tau}} \sim N(0,1)$

That $\sigma\left(\widetilde{W}(T) - \widetilde{W}(t_n)\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t_n) = \sigma\sqrt{\tau}Z + \left(r - \frac{1}{2}\sigma^2\right)\tau$

Using the integral of the PDF $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, we obtain：$c_n(t_n, x) = \int_{-\infty}^{\infty} e^{\{-r\tau\}} \left(xe^{\left\{(r - 0.5\sigma^2)\tau + \sigma\sqrt{\{\tau\}z}\right\}} - K\right)^+ \phi(z)dz$

where $z > \frac{\ln\left(\frac{K}{x}\right) - (r - 0.5\sigma^2\tau)}{\sigma\sqrt{\tau}} = -d_-$, $c_n(t_n, x) = \int_{-d_-}^{\infty} e^{\{-r\tau\}} \left(xe^{\left\{(r - 0.5\sigma^2)\tau + \sigma\sqrt{\{\tau\}z}\right\}} - K\right)^+ \phi(z)dz$

$c_n(t_n, x) = \int_{-d_-}^{\infty} e^{\{-r\tau\}}\, xe^{\left\{(r - 0.5\sigma^2)\tau + \sigma\sqrt{\{\tau\}z}\right\}}\phi(z)dz - \int_{-d}^{\infty} e^{\{-r\tau\}} K\phi(z)dz$,and use $u = z - \sigma\sqrt{\tau}$

8.5.1 Underiying Asset Pays No Dividends

$c_n\ (t,x)$ also satisfies the Black-Scholes-Merton differential equation

$$\frac{\partial}{\partial t}\boldsymbol{c_n}\ (\boldsymbol{t},\boldsymbol{x}) + \boldsymbol{rx}\frac{\partial}{\partial x}\boldsymbol{c_n}\ (\boldsymbol{t},\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\delta}^2\boldsymbol{x}^2\frac{\partial^2}{\partial x^2}\boldsymbol{c_n}\ (\boldsymbol{t},\boldsymbol{x}) = \boldsymbol{rc_n}\ (\boldsymbol{t},\boldsymbol{x}), \boldsymbol{t_n}\ \leq \boldsymbol{t}\ \leq \boldsymbol{T}, \mathbf{x} \geq \mathbf{0} \qquad \textbf{(8.5.16)}$$

and the terminal condition $\boldsymbol{c_n}\ (\boldsymbol{t},\boldsymbol{x}) = (\boldsymbol{x} = \boldsymbol{K})^+, \boldsymbol{x} \geq \mathbf{0}$ **(8.5.17)**

To show $c_n\ (t,x)$ convexity in x, we show that, whenever $0 \leq x_1 \leq x_2$ and $0 \leq \lambda \leq 1$,

we have $\boldsymbol{c_n}\ (\boldsymbol{t_n}, (\mathbf{1}-\boldsymbol{\lambda})\boldsymbol{x_1} + \boldsymbol{\lambda}\boldsymbol{x_2}) \leq (\mathbf{1}-\boldsymbol{\lambda})\boldsymbol{c_n}(\boldsymbol{t_n},\boldsymbol{x_1}) + \boldsymbol{\lambda}\boldsymbol{c_n}(\boldsymbol{t_n},\boldsymbol{x_2})$ **(8.5.18)**

First, observe that $(ax-K)^+$ is convex in x for any a.

Thus, $\left(x \exp\left\{\sigma\left(\widetilde{W}(T)-\widetilde{W}(t_n)\right) + \left(r-\frac{1}{2}\sigma^2\right)(T-t_n)\right\} - K\right)^+$ is also convex in x

8.5.1 Underiying Asset Pays No Dividends

$$C_n\,(t_n, (1-\lambda)x_1 + \lambda x_2)$$

$$= \widetilde{E}\left[\left(e^{-r(T-t_n)}\big((1-\lambda)x_1 + \lambda x_2\big)\, exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t_n)\right) + \left(r - \frac{1}{2}\sigma^2\right)\right\} - K\right)^+\right]$$

$$\leq (1-\lambda)\widetilde{E}\left[\left(e^{-r(T-t_n)}(x_1\, exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t_n)\right) + \left(r - \frac{1}{2}\sigma^2\right)\right\} - K\right)^+\right]$$

$$+ \lambda\widetilde{E}\left[\left(e^{-r(T-t_n)}(x_2\, exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t_n)\right) + \left(r - \frac{1}{2}\sigma^2\right)\right\} - K\right)^+\right]$$

$$= (1-\lambda)c_n(t_n, x_1) + \lambda c_n(t_n, x_2) \qquad \textbf{(8.5.19)}$$

8.5.1 Underiying Asset Pays No Dividends

Immediately before the dividend at t_n, the holder can exercise for $S(t_n -) - K$ or decline,

holding an option worth $c_n(t_n, (1-\alpha_n)S(t_n-))$ as the price falls to $S(t_n) = (1-a_n)S\left(t_{\{n\}} -\right)$.

Optimally, exercise if $S(t_n -) - K > c_n(t_n, (1-\alpha_n)S(t_n-))$, decline if less, and indifferent if equal.

Therefore, the pre-dividend value at t_n is $h_n(S(t_n) -)$

with $\boldsymbol{h_n(x) = \max\{x - K, c_n\left(t_n, \left(1 - a_{\{n\}}\right)x\right)\}, x \geq 0}$ **(8.5.20).**

8.5.1 Underiying Asset Pays No Dividends

Since $c_n(t_n, (1-\alpha_n)x) \geq 0$ for all $x \geq 0$, we have $h_n(x) \geq 0$

Also, $c_n(t_n, (1-\alpha_n)0) = 0$ that $h_n(x) = 0$. Thus, $h_n(x)$ satisfies Lemma 8.5.1.

To establish convexity, recall from (8.5.18) that $c_n\,(t, x)$ is convex in x.

For $0 \leq \lambda \leq 1$, replacing x with $(1-\alpha_n)x$ into (8.5.18) to obtain

$$\mathrm{c_n}\,(\mathrm{t_n}, (1-\alpha_n)((1-\lambda)\mathrm{x}_1 + \lambda \mathrm{x}_2)) \leq (1-\lambda)\mathrm{c_n}(\mathrm{t_n}, (1-\alpha_n)\mathrm{x}_1) + \lambda \mathrm{c_n}(\mathrm{t_n}, (1-\alpha_n)\mathrm{x}_2)$$

This shows that $c_n(t, (1-\alpha_n)x)$ is a convex function of x.

Since the maximum of convex functions is convex, $h_n(x)$ is convex.

8.5.1 Underiying Asset Pays No Dividends

For $t_{n-1} \leq t \leq t_n$, the holder can exercise at any $u \epsilon [t, t_n)$ to receive $S(u) - K$.

If held until t_n , the pre-dividend value is $h_n\big(S(t_n -)\big)$.

Thus, the original call expiring atT is equivalent to an American option expiring just before t_n

with payoff $h_{n(S(t_n—))}$.

Since the asset follows a geometric Brownian motion between dividends(t_{n-1} *to* t_n),

Lemma 8.5.1 implies $e^{-rt}h_n(S(t))$ is a ""submartingale.

Specifically, $\tilde{E}\big[e^{-r(u-t)}h_n(S(u)\big|F(t)\big] \geq h_n\big(S(t)\big), t_{n-1} \leq t \leq u \leq t_n$

letting $u \uparrow t_n$, we obtain $\boldsymbol{\widetilde{E}\big[e^{-r(t_n-t)}h_n\big(S(t_n -)\big)\big|F(t)\big] \geq h_n\big(S(t)\big)}$ **(8.5.21)**

By the definition of $h_n(x)$, $\boldsymbol{h_n\big(S(t)\big) \geq S(t) - K}$ **(8.5.22)**

8.5.1 Underiying Asset Pays No Dividends

This implies the European value at t_n with payoff $h_n\big(S(t_n -)\big)$ exceeds the American intrinsic value. Consequently, early exercise before t_n is worthless, and the American call is equivalent to this European call.

Since $S(t)$ is Markov, there exists a function $c_{n-1}\,(t, x)$ for the period between dividends such that:

$$\boldsymbol{c_{n-1}\,(t, S(t)) = \widetilde{E}\,[e^{-r(t_n-t)} h_n\big(S(t_n -)\big)|F(t)]} \qquad \textbf{(8.5.23)}$$

8.5.1 Underiying Asset Pays No Dividends

The function $c_{n-1}\,(t, x)$ is determined by the conditional expectation in (8.5.23) given $S(t) \;=\; x$.

For $t \;=\; t_{n-1}$, using (8.5.10) yields:

$\boldsymbol{c_{n-1}\,(t_{n-1}, x) =}$

$$\widetilde{E}[e^{-r(t_n - t_{n-1})} h_n(x \exp\left\{\sigma\left(\widetilde{W}(t_n) - \widetilde{W}(t_{n-1})\right) + \left(r - \tfrac{1}{2}\sigma^2\right)(t_n - t_{n-1})\right\})] \quad \textbf{(8.5.25)}$$

$c_{n-1}\,(t, x)$ also satisfies the Black-Scholes-Merton differential equation:

$$\frac{\partial}{\partial t} c_{n-1}\,(t, x) + rx\frac{\partial}{\partial x} c_{n-1}\,(t, x) + \frac{1}{2}\delta^2 x^2 \frac{\partial^2}{\partial x^2} c_{n-1}\,(t, x) = rc_{n-1}\,(t, x), t_{n-1} \;\le t \;\le \mathrm{t_n}, \mathrm{x} \ge 0 \quad \textbf{(8.5.26)}$$

and the terminal condition $\boldsymbol{c_{n-1}\,(t_n, x) = h_n(t_n, x), x \ge 0}$ **(8.5.27)**

8.5.1 Underiying Asset Pays No Dividends

We repeat by defining $h_{n-1}(x) = \max\{x - K, c_{n-1}(t_{n-1}, (1 - a_{n-1})x)\}, x \geq 0$, which satisfies Lemma 8.5.1. This yields an algorithm for American calls with dividends at $t_1, t_2, \ldots, t_n$. Recursively solve the PDE (8.5.28) for $j = n, n-1, \ldots, 0$, the partial differential equation

$$\frac{\partial}{\partial t} c_{j-1}(t,x) + rx\frac{\partial}{\partial x} c_{j-1}(t,x) + \frac{1}{2}\delta^2 x^2 \frac{\partial^2}{\partial x^2} c_{j-1}(t,x) = rc_{j-1}(t,x), t_{j-1} \leq t \leq t_j, x \geq 0 \quad \textbf{(8.5.28)}$$

with the terminal condition $c_{j-1}(t_j, x) = h_j(x), x \geq 0 \quad \textbf{(8.5.29)}$

8.5.1 Underiying Asset Pays No Dividends

Initialize withc_n (t, x) and $h_„(x)$ from (8.5.14) and (8.5.20); the subsequent $h_{j-1}(x)$ is given by

$$\boldsymbol{h_{j-1}(x) = \max\{x - K, c_{j-1}(t_{j-1}, (1 - a_{j-1})x)\}, x \geq 0} \quad \textbf{(8.5.30).}$$

For $t_{j-1} \leq t \leq \mathrm{t}_j$, if $S(t) = x$, then $\boldsymbol{c_{j-1}(t, x)}$ is the American call price.

Within each interval $[t_{j-1}, t_j)$,effectively acting as a European call expiring at t_j.

Optimal exercise occurs at the earliest pre-dividend t_j where $S(t_j -)$— $K > c_j(t_j, (1 - a_j)S(t_j-))$

If no such t_j exists, exercise at T if $S(T) > K$, otherwise expire.