7.5~7.5.2

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7.5 Asian Options

An Asian option's payoff is based on the average price of the underlying asset over a specified period. This period can span the entire duration from issuance to expiration, or it can be a shorter interval within that timeframe.

The average from continuous sampling $\boldsymbol{\frac{1}{T}\int_0^T S(t)dt}$, or from discrete sampling $\boldsymbol{\frac{1}{m}\sum_{j=1}^{m} S(t_j)}$

where $0 < t_1 < t_2 \ldots . < t_m = T$

Since no closed-form solution exists, we will now derive the partial differential equation (PDE) for pricing the Asian option.

7.5.1 Fixed-Strike Asian Call

We begin with a geometric Brownian motion S(t) given by $\boldsymbol{dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)}$ $\quad \mathbf{(7.5.1)}$

where $\widetilde{W}$(t), $0 \le t \le T$, is a Brownian motion under the risk-neutral measure $\widetilde{P}$.

Consider a fixed-strike Asian call whose payoff at time T is $\boldsymbol{V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+}$ $\quad \mathbf{(7.5.2)}$

where the strike price K is a nonnegative constant.

The price T of this call at times t by the risk-neutral pricing formula：

$\boldsymbol{V(t) = \widetilde{E}\left[e^{-r(T-t)}V(T)\middle|F(t)\right], 0 \le t \le T}$ $\mathbf{(7.5.3)}$

Multiplying both sides by e^{-rt} yields：

$\boldsymbol{e^{-rt}V(t) = \widetilde{E}\left[e^{-rT}V(T)\middle|F(t)\right], 0 \le t \le T}$, is a martingale under $\widetilde{P}$.

7.5.2 Augmentation of the State

The payoff V(T) in (7.5.2) is path-dependent.

Therefore, we cannot claim that V(t) is a function of only t and S(t), as this violates the Markov property.

To overcome this difficulty, we augment the state S(t) by defining a second process

$$\mathbf{Y}(\boldsymbol{t}) = \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{S}(\boldsymbol{u})\boldsymbol{du} \quad (\mathbf{7.5.4})$$

The stochastic differential equation for Y (t) is thus

$$\boldsymbol{dY}(\boldsymbol{t}) = \boldsymbol{S}(\boldsymbol{t})\boldsymbol{dt} \quad (\mathbf{7.5.5})$$

7.5.2 Augmentation of the State

Because the processes S(t) and Y(t) are governed by the stochastic differential equations (7.5.1) and (7.5.5), respectively, the pair (S(t),Y(t))$(S(t),Y(t))$ thus forms a two-dimensional Markov process by Corollary 6.3.2.

Furthermore, the call payoff V (T)) is a function of T and the final value-(S(T),Y(T))-of this process.

Indeed, V(T) depends only on T and Y (T), by the formula： $\boldsymbol{V(T)=\left(\frac{1}{T}Y(T)-K\right)^{+}} \qquad \boldsymbol{(7.5.6)}$

This implies that there must exist some function v(t, z,y) such that the Asian call price (7.5.3) is given as

$$\boldsymbol{V(t,S(t),Y(t)) \;=\widetilde{E}\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T)-K\right)\middle|F(t)\right]=\widetilde{E}\left[e^{-r(T-t)}V(T)\middle|F(t)\right],0\le t\le T} \qquad \boldsymbol{(7.5.7)}$$

The function $v(t,x,y)$ satisfies a partial differential equation.

In addition, for numerical solution, it is usually necessary to specify the asymptotic behavior of $v(t,x,y)$ as $x\rightarrow\infty$and as $y\rightarrow\pm\infty$

Theorem 7.5.1. The Asian call price function v(t,x,y) of (7.5.7) satisfies the partial differential equation

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y), 0 \leq t \leq T, x \geq 0, y \in R \qquad (7.5.8)$$

and the boundarıj conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+, 0 \leq t \leq T, y \in R \qquad (7.5.9)$$

$$\lim_{y \downarrow -\infty} v(t,x,y) = 0, 0 \leq t \leq T, y \in R \qquad (7.5.10)$$

$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+, x \geq 0, y \in R \qquad (7.5.11)$$

PROOF :

$\mathbf{dS(t)}\boldsymbol{dY(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t) * S(t)dt}$

$\mathbf{dY(t)}\boldsymbol{dY(t) = S(t)dt * S(t)dt}$

Using the SDE (7.5.1) and (7.5.5) and noting that $dS(t)dY\,(t) = 0 \; and \; dY(t)dY(t) \;=\; 0$,

we take the differential of the $\tilde{P}$ -martingale $e^{-rt}V(t) = e^{-rt}v(t, S(t), Y\,(t))$.

$dt \times dt = 0$

$dt \times d\widetilde{W} = 0$

$d\widetilde{W} \times d\widetilde{W} = dt$

This differential is $d\left(e^{-rt}v\big(t, S(t), Y(t)\big)\right) = e^{-rt}\left[-rvdt + v_t dt + v_x dS + v_y dY + \frac{1}{2}v_{xx}dSdS\right]$

$$= e^{-rt}\left[-rv + v_t + rSv_x + Sv_y + \frac{1}{2}\sigma^2 S^2 v_{xx}\right]dt + e^{-rt}\sigma S v_x d\widetilde{W}(t) \quad (7.5.12)$$

In order for this to be a martingale, the dt term must be zero, which implies

$$v_t(t, S(t), Y(t)) + rS(t)v_x\big(t, S(t), Y(t)\big) + S(t)v_y\big(t, S(t), Y(t)\big) + \frac{1}{2}\sigma^2 S^2(t)v_{xx}(t, S(t), Y(t))$$

$$= rv(t, S(t), Y(t))$$

Replacing S(t) by the dummy variable x and Y (t) by the dummy variable y, we obtain (7.5.8).

PROOF：

We note that S(t) must always be nonnegative, and so (7.5.8) holds for x ≥ 0.

If $S(u) = 0$ for all $u \in [t,T]$, $S(t) = 0 \; and \; Y(t) = y$ for some value of t, then $Y\,(u)$ is constant on $[t,T]$.

Therefore, $Y(T) = y$, and the value of the Asian call at time t is $\left(\boldsymbol{\frac{y}{T} - K}\right)^{+}$, discounted from T back to t.

This gives us the boundary condition $\boldsymbol{v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^{+}, 0 \leq t \leq T, y \in R\ (7.5.9)}$.

In contrast, it is not the case that if $Y(t) = 0$ for some time t, then $Y(u) = 0 \; for \; all \; u > 0$.

Therefore, we can not easily determine the value of $v(t,x,0)$, and we do not provide a condition on the boundary y = 0.

Indeed, at least mathematically there is no problem with allowing to be negative. If at time t we set $Y\,(t) = y$, then $Y(T)$ is defined by (7.5.5). In integrated form, this formula is

$$\mathbf{Y}(\boldsymbol{T}) = \boldsymbol{y} + \int_{\boldsymbol{t}}^{\boldsymbol{T}} \boldsymbol{S(u)du} \qquad \mathbf{(7.5.13)}$$

PROOF：

Even if y is negative, this makes sense, and in this case we could still have $Y(T) > 0$ or even $\frac{1}{T}Y(T) — K > 0$, so that the call expires in the money.

When S(t) and Y(t) are governed by the differential equations (7.5.1) and (7.5.5), although S(t) is required to be nonnegative, there is no reason to require that Y(t) also be nonnegative.

If $Y(t) = y, S(t) = x$, and holding x fixed we let $\boldsymbol{y} > -\infty$, then Y(T) approaches $-\infty$,
The probability that the call expires in the money approaches gero, and the option price approaches zero.

The natural boundary for y is $y = -\infty$, and the boundary condition there is

$$\boldsymbol{\lim_{y \downarrow -\infty} v(t,x,y) = 0, 0 \le t \le T, y \in R} \qquad \boldsymbol{(7.5.10)}$$

The boundary condition $\boldsymbol{v(T,x,y) = \left(\frac{y}{T} - K\right)^{+}, x \ge 0, y \in R}$ $\boldsymbol{(7.5.11)}$ is just the payoff of the call.

Remark 7.5.2

After we set the dt term in (7.5.12) equal to zero, we see that

$$d\left(e^{-rt}v\big(t,S(t),Y(t)\big)\right)=e^{-rt}\sigma S(t)v_x(t,S(t),Y(t))d\widetilde{W}(t) \qquad (7.5.14)$$

Following (5.2.27), the discounted value of a portfolio holding Δ(t) shares of the underlying asset at each time t is given by：$d(e^{-rt}X(t))=e^{-rt}\sigma S(t)\Delta(t)d\widetilde{W}(t)$ (7.5.15)

To hedge a short position in an Asian call option, one should equate these two differentials, which yields the Delta-hedging formula：$\Delta(t)=v_x(t,S(t),Y(t))$