6.3 ~ 6.5

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6.3 The Markov Property

Consider the stochastic differential equation (6.2.1) => $dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u).$

Let $0 \leq t \leq T$ be given, and let h(y) be a Borel-measurable function.

Denote by $\boldsymbol{g(t,x) = E^{t,x}h\big(X(T)\big)}$, the expectation of $h\big(X(T)\big)$

where $X(T)$ is the solution to (6.2.1) with initial condition $X(t) = x$.

(assume that $E^{t,x}h(X(T)) < \infty$)

Note：there is **nothing random** about $g(t,x)$ => Because of expectation

it is an Borel-measurable function of the two dummy variables t and x

6.3 The Markov Property

If not have an explicit formula for the distribution of $X(T)$,One way to do this would be to use the

Euler method :choose a small positive step size δ and beginning at $\boldsymbol{X(T) = x}$

Then set $X(t+\delta) = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\epsilon_1$

δ is chosen so that $(\frac{\mathrm{T}-t}{\delta})$ is an integer

Next step $X(t+2\delta) = x(t+\delta) + \beta(t+\delta, X(t+\delta))\delta + \gamma(t+\delta, X(t+\delta))\sqrt{\delta}\epsilon_2$

Where $\epsilon_1 \& \epsilon_2$ are a standard normal random variable and independent

By this device, one eventually determines a value for $X(T)$ (corresponding to one ω)

Now repeat this process many times and compute the average of h(X(T)) over all

these simulations to get an **approximate** value for $g(t,x)$.

Note： if one were to begin with a different time t and initial value x, one would get a different answer . This dependence on t and z is emphasized by the notation $E^{t,x}$ in (6.3.1).

Theorem 6.3.1.

Let $X(u)$, $\mathrm{u} \geq 0$, be a solution to the stochastic differential equation (6.2.1) ,

with initial condition given at time 0. Then, for $0 \leq t \leq T$

$$\Rightarrow \ \boldsymbol{E}\big[\boldsymbol{h}\big(\boldsymbol{X}(\boldsymbol{T})\big)\big|\boldsymbol{F}(\boldsymbol{t})\big] = \boldsymbol{g}\big(\boldsymbol{t}, \boldsymbol{X}(\boldsymbol{t})\big) \quad (\mathbf{6.3.2})$$

While the details of the proof of Theorem 6.3.1 are quite technical and will not be given

But we can the process X (u) begins at time zero, being generated by the stochastic differential

equation (6.2.1), and one watches it up to time t.

Now based on this information, to compute the conditional expectation of h(X(T)), where T > t.

Theorem 6.3.1.

Then one should pretend that the process is **starting at time t** at its current position, simulate the SDE from t to T, obtain a new path, and use the resulting $X(T)$ to compute the expected value of $h(X(T))$, which gives $g(t,x)$.

In other words, **replace $\boldsymbol{X(t)}$ by a dummy x** in order to hold it constant, compute $g(t,x) = E^{t,x}h\big(X(T)\big)$ and after computing this function put the random variable $X(t)$ back in place of the dummy x.

Note： $X(T)$ depends on $X(t)$ and Brownian increments over $[t,T]$, which are independent of F(t). => It ensures that $E[h(X(T)) \mid F(t)] = E[h(X(T)) \mid X(t)]$

$$= g(t, X(t))$$

6.4 Partial Differential Equations

Feynman-Kac Theorem：

relates stochastic differential equations and partial differential equations.

When this partial differential equation is solved, it yields the function $g(t, x)$ defined in (6.3.1).

Two method to give $g(t, x)$：

Euler method：converges slowly & provides only a single function value at a (t, x).

By solving equation (6.4.1)：faster convergence & obtain all values of (t, x) function

Theorem 6.4.1 (Feynman-Kac)

Consider the stochastic differential equation $\Longrightarrow$ $$dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u).$$

Let h(y) be a Borel-measurable function. Fix T > 0, and let t ϵ [0,T] be given.

Define the function $\boldsymbol{g(t,x) = E^{t,x} h\big(X(T)\big)}$ (6.3.1)

(assume that $E^{t,x}h(X(T)) < \infty$ for all t and x)

Then $g(t,x)$ satisfies the PDE : $\boldsymbol{g_t(t,x) + \beta(t,x) g_x(t,x) + \frac{1}{2}\gamma^2(t,x) g_{xx}(t,x) = 0}$ (6.4.1)

and the terminal condition : $\boldsymbol{g(T,x) = h(x)}$ for all x (6.4.2)

Lemma 6.4.2.

The proof of the Feynman-Kac Theorem depends on the following lemma.

Let X(u) be the solution to SDE (6.2.1) with initial condition given at 0.

Let h(y) be a Borel-measurable function, fix T > 0, and let $g(t,x)$ be given by (6.3.1).

According to the previous Theorem 6.3.1 ⇒ $E\left[h\left(X(T)\right)\middle|F(t)\right] = g\left(t,X(t)\right)$

Then we can know the stochastic process $\boldsymbol{g\left(t,X(t)\right), 0 \leq t \leq T}$ is a martingale.

PROOF: Let $0 \leq s \leq t \leq T$ be given. Theorem 6.3.1 implies

$$E\left[h\left(X(T)\right)\middle|F(s)\right] = g\left(s,X(s)\right) \quad / \quad E\left[h\left(X(T)\right)\middle|F(t)\right] = g\left(t,X(t)\right)$$

If use repeated conditional expected values：

$$\boldsymbol{E\left[h\left(X(T)\right)\middle|F(s)\right] = E\left[E[h\left(X(T)\right)\middle|F(t)]|F(s)\right] = E\left[h\left(X(T)\right)\middle|F(s)\right] = g\left(s,X(s)\right)}$$

OUTLINE OF PROOF OF THEOREM 6.4.1

Let X(t) be the solution to SDE (6.2.1) starting at 0

From Lemma 6.4.2, $g\big(t, X(t)\big)$ is a martingale,so its drift term must be zero.

Therefore, the coefficient of dt in Itô's Lemma must be 0.

Apply Itô's Lemma to $g\big(t, X(t)\big)$, using the SDE (6.2.1) for substitution

$$\boldsymbol{dg\big(t, X(t)\big) = g_t dt + g_x dX + \frac{1}{2} g_{xx} dX dX} \longrightarrow \begin{array}{l} \boldsymbol{dX(t) = \beta(t, X(t))dt + \gamma\big(t, X(t)\big)dW(t)} \\ \boldsymbol{dX(t)dX(t) = \gamma^2\big(t, X(t)\big)dt} \end{array}$$

$$\boldsymbol{= g_t dt + \beta g_x dt + \gamma g_x dW + \frac{1}{2}\gamma^2 g_{xx} dt = \left[g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx}\right] dt + \gamma g_x dW}$$

Setting the dt term to zero and putting back the argument $g\big(t, X(t)\big)$

we obtain $\boldsymbol{g_t\big(t, X(t)\big) + \beta\big(t, X(t)\big) g_x\big(t, X(t)\big) + \frac{1}{2}\gamma^2\big(t, X(t)\big) g_{xx}\big(t, X(t)\big) = 0}$

along every path of X： $\boldsymbol{g_t(t, x) + \beta(t, x) g_x(t, x) + \frac{1}{2}\gamma^2(t, x) g_{xx}(t, x) = 0}$

Theorem 6.4.3 (Discounted Feynman-Kac)

Under the same assumptions and conditions on the SDE as before

Defîne the function： $\mathbf{f}(\boldsymbol{t},\boldsymbol{x}) = \boldsymbol{E}^{\boldsymbol{t},\boldsymbol{x}}[\boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})}\boldsymbol{h}(\boldsymbol{X}(\boldsymbol{T}))]$ (6.4.3)

(assume that $E^{t,x}h(X(T)) < \infty$ for all t and x)

Then f(t,x) satisfies the PDE： $\boldsymbol{f}_{\boldsymbol{t}}(\boldsymbol{t},\boldsymbol{x}) + \boldsymbol{\beta}(\boldsymbol{t},\boldsymbol{x})\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{t},\boldsymbol{x}) + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\gamma}^{\mathbf{2}}(\boldsymbol{t},\boldsymbol{x})\boldsymbol{f}_{\boldsymbol{xx}}(\boldsymbol{t},\boldsymbol{x}) = \boldsymbol{r}\boldsymbol{f}(\boldsymbol{t},\boldsymbol{x})$ (6.4.4)

and the terminal condition： $\boldsymbol{f}(\boldsymbol{T},\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})$ for all x (6.4.5)

But the difference is $\mathbf{f}(\boldsymbol{t},\boldsymbol{X}(\boldsymbol{t}))$ is not a martgale

Theorem 6.4.3 (Discounted Feynman-Kac)

OUTLINE OF PROOF:

Let $X(t)$ be the solution to the SDE (6.2.1) starting at t = 0 & $\mathrm{f}(t, X(t)) = E\,[e^{-r(T-t)}h\big(X(T)\big)|F(t)]$

If $0 \le s \le t \le T$, then $\boldsymbol{E[f(t, X(t))|F(s)] = E\big[E[e^{-r(T-t)}h\big(X(T)\big)\big|F(t)]|F(s)\big]}$

$$\boldsymbol{= E\big[e^{-r(T-t)}h\big(X(T)\big)\big|F(s)\big]}$$

differing discount terms

which is not the same as $\boldsymbol{\mathrm{f}(s, X(s)) = E\,[e^{-r(T-s)}h\big(X(T)\big)|F(s)]}$

In order to get the martingale property from iterated conditioning

The variable being estimated can't depend on time t.

To achieve this, we “complete the discounting,” observing that

$$\boldsymbol{e^{-rt}\mathrm{f}(t, X(t)) = E\,[e^{-r\mathrm{T}}h\big(X(T)\big)|F(t)]} \Rightarrow \boldsymbol{E[e^{-rt}f(t, X(t))|F(s)] = E\big[E[e^{-r\mathrm{T}}h\big(X(T)\big)\big|F(t)]|F(s)\big]}$$

$$\boldsymbol{= E\big[E[e^{-r\mathrm{T}}h\big(X(T)\big)\big|F(s)\big] = e^{-rs}\mathrm{f}(s, X(s))}$$

Theorem 6.4.3 (Discounted Feynman-Kac)

We may now apply iterated conditioning to show that $e^{-rt}\mathrm{f}(t, X(t))$ is a martingale.

The differential of this martingale is

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t)$$

$$dX(t)dX(t) = \gamma^2(t, X(t))dt$$

$$d\left(e^{-rt}\mathrm{f}(t, X(t))\right) = e^{-rt}\left[-rf dt + f_t dt + f_x dX + \frac{1}{2} f_{xx} dX dX\right]$$

$$= e^{-rt}\left[-rf + f_t + \beta f_x dt + \frac{1}{2}\gamma^2 f_{xx}\right] dt + e^{-rt}\gamma f_x dW$$

Setting the dt term to zero and putting back the argument $e^{-rt}\mathrm{f}(t, X(t))$

we obtain $f_t(t,x) + \beta(t,x) f_x(t,x) + \frac{1}{2}\gamma^2(t,x) f_{xx}(t,x) = rf(t,x)$ (6.4.4)

along every path of X : $g_t(t,x) + \beta(t,x) g_x(t,x) + \frac{1}{2}\gamma^2(t,x) g_{xx}(t,x) = 0$

Example 6.4.4 (Options on a geometric Brownian motion)

Let h(S(T)) be the payoff at time T of a derivative security

whose underlying asset is the geometric Brownian motion $\Rightarrow dS(u) = \alpha S(u)du + \sigma S(u)dW(u)$

rewriting under the risk-neutral measure $\tilde{P}$ $\Rightarrow dS(u) = rS(u)du + \sigma S(u)d\widetilde{W}(u)$

By the risk-neutral pricing formula (5.2.31) , the derivative's price at time t is： stock price is Markov

$$\boldsymbol{V(t) = \widetilde{E}[e^{-r(T-t)}h(S(T))|F(t)]} \qquad (6.4.8)$$

payoff is a function of the stock price

We can express the price as a function $v(t,x)$, then substitute x=$S(t)$ to get $V(t) = v(t, S(t))$

$v(t,x)$ must satisfy the PDE equation (6.4.4)

$$\boldsymbol{d\left(e^{-rt}v(t,S(t))\right) = e^{-rt}\left[-rvdt + v_t dt + v_s dS + \frac{1}{2}v_{ss}dSdS\right]}$$

$$\boldsymbol{dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)}$$

$$\boldsymbol{dS(t)dS(t) = \sigma^2 S^2(t)dt}$$

$$\boldsymbol{= e^{-rt}\left[-rv + v_t + \gamma S(t)v_S dt + \frac{1}{2}\sigma^2 S^2(t)v_{SS}\right]dt + e^{-rt}\,\sigma S(t)f_S dW}$$

along every path of S： $\boldsymbol{v_t(t,x) + \gamma x v_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)}$

Example 6.4.4 (Options on a geometric Brownian motion)

If σ depends on time and stock price (i.e.$\sigma(t,x)$),then the stock price would no longer be a geometric Brownian motion and the Black-Scholes-Merton formula would no longer apply.

However, the option price can still be found by solving PDE (6.4.9),

with the constant $\boldsymbol{\sigma^2}$ replaced by $\boldsymbol{\sigma^2(t,x)}$: $\boldsymbol{v_t(t,x)+\gamma x v_x(t,x)+\frac{1}{2}\sigma^2(t,x)x^2 v_{xx}(t,x)=rv(t,x)}$

It has been observed in markets that if one assumes a constant volatility

(6.4.9) lets us back out the σ that makes the model price match the market

this σ is the implied volatility. For a given maturity, implied volatility varies by strike and

typically forms a convex function , one refers to this phenomenon as the volatility smile

Example 6.4.4 (Options on a geometric Brownian motion)

To explain the volatility smile, we propose a simple model with non-constant volatility

The constant elasticity of variance (CEV) model, in which $\boldsymbol{\sigma(t,x) = \sigma x^{\delta-1}}$

(depends on x but not t and $\delta \in (0,1)$)

It models volatility as $\sigma S^{\delta-1}$, a decreasing function of price (vol falls as S rises)

Under the risk-neutral measure $\tilde{P}$, the stock SDE follows

$$\boldsymbol{dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t) \Rightarrow \sigma = \sigma x^{\delta-1}, x = S(t) \Rightarrow dS(t) = rS(t)dt + \sigma S^{\delta}(t)d\widetilde{W}(t)}$$

This explains why, for a fixed maturity, implied volatility differs by strike

To capture variation across strikes and expiries

let volatility depend on time and price,$\boldsymbol{\sigma(t,x)}$—the “volatility surface.”

6.5 Interest Rate Models

The simplest models for fixed income markets begin with a SDE for the interest rate

$$\boldsymbol{dR(t) = \beta(t, R(t))\, dt + \gamma(t, R(t))\, d\widetilde{W}(t)}$$

Pricing under $\tilde{P}$ ensures discounted asset prices are martingales (i.e. no arbitrage)

Models for the interest rate R(t) represents the short-term interest rate, so these are called

short-rate models , they describe how R(t) evolves randomly over time.

If the model has only one stochastic driver (e.g. a single Brownian motion), it is a *one-factor model*. Such models can only produce parallel shifts of the yield curve and cannot capture changes in slope, curvature, or other complex shape variations.

6.5 Interest Rate Models

Now the discount process is as given in (5.2.17) $\boldsymbol{D(\mathrm{t}) = e^{-\int_0^t R(s)ds}}$

and we denote the money market account price process to be $\boldsymbol{\frac{1}{D(t)} = e^{\int_0^t R(s)ds}}$

As discussed following (5.2.18), we have the differential formulas

$$\mathbf{d}\boldsymbol{D}(\mathbf{t}) = -\mathbf{R(t)D(t)dt}\,,\, \mathbf{d}\left(\frac{\mathbf{1}}{\boldsymbol{D(t)}}\right) = \frac{\boldsymbol{R(t)}}{\boldsymbol{D(t)}}\boldsymbol{dt}$$

For a zero-coupon bond with face value 1 and maturity T, the risk-neutral pricing formula (5.2.30) implies

that, under the risk-neutral measure, the discounted bond price is a martingale.

In other words, for $0 \leq t \leq T$, the bond price at time t $B(t, T)$ must satisfy

$$\boldsymbol{D(\mathrm{t})B(t,T) = \widetilde{E}[D(T)|F(t)]} \text{ (payoff=1) } \Rightarrow \boldsymbol{D(t) \Rightarrow B(t,T) = \widetilde{E}\left[\frac{D(T)}{D(t)}\middle|F(t)\right] = \widetilde{E}\left[e^{-\int_s^T R(s)ds}\middle|F(t)\right]}$$

Define the yield between t and T to be $\boldsymbol{Y(\mathrm{t}) = -\frac{1}{T-t}\log B(t,T)}$ or, equivalently, $\boldsymbol{B(t,T) = e^{-Y(t,T)(T-t)}}$

6.5 Interest Rate Models

Since R is given by a SDE, it is a Markov process and we must have $B(t,T) = f(t, R(t))$

To find PDE for the unknown function $f(t,r)$, we find a martingale, take its differential,

and set the dt term equal to zero.

Then the martingale in this case is $D(t)B(t,T) \;=\; D(t)\, f(t, R(t))$. Its differential is

$$\mathrm{d}\Big(\mathrm{D(t)f}\big(\mathrm{t,R(t)}\big)\Big) = \mathrm{f}\big(\mathrm{t,R(t)}\big)\mathrm{dD(t)} + \mathrm{D(t)df}\big(\mathrm{t,R(t)}\big) = \mathrm{D(t)}\left[-\mathrm{Rfdt} + f_t dt + f_r dR + \frac{1}{2} f_{rr} dRdR\right]$$

$$= \mathrm{D(t)}[-\mathrm{Rf} + f_t + \beta f_r + \frac{1}{2}\gamma^2 f_{rr}]dt + D(t)\gamma f_r d\widetilde{W}$$

$\boldsymbol{dD(t) = -R(t)D(t)dt}$

$\boldsymbol{dR(t)=\beta(t,R(t))\, dt+\gamma(t,R(t))\, d\widetilde{W}(t)}$

$\boldsymbol{dRdR(t)=\gamma^2(t,R(t))\, dt}$

Let dt = 0, we obtain the PDE $f_t(t,r) + \beta(t,r) f_r(t,r) + \frac{1}{2}\gamma^2(t,r) f_{rr}(t,r) = rf(t,r)$ (6.5.4)

We also have the terminal condition $\mathrm{f}(\mathrm{T}, r) = 1$ for all r (bond at maturity is its face value 1)

Example 6.5.1 (Hull-White interest rote model)

In the Hull-White model , the evolution of the interest rate is given by

$$\boldsymbol{dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)\, d\widetilde{W}(t)}$$

(where $a(t)$, $b(t)$, and $\sigma(t)$ are nonrandom positive functions of time)

The partial differential equation (6.5.4) for the zero-coupon bond price becomes

$$\mathrm{d}\Big(\mathrm{D(t)f}\big(\mathrm{t,R(t)}\big)\Big) = \mathrm{f}\big(\mathrm{t,R(t)}\big)\mathrm{dD(t)} + \mathrm{D(t)df}\big(\mathrm{t,R(t)}\big) = \mathrm{D(t)}\left[-\mathrm{Rfdt} + f_t dt + f_r dR + \frac{1}{2} f_{rr} dR dR\right]$$

$$= \mathrm{D(t)}[-Rf + f_t + = (a(t) - b(t)R(t))f_r + \tfrac{1}{2}\sigma^2 f_{rr}]dt + D(t)\sigma(t)f_r d\widetilde{W}$$

$\boldsymbol{dR(t)} = \boldsymbol{(a(t) - b(t)R(t))dt + \sigma(t)\, d\widetilde{W}(t)}$

$\boldsymbol{dRdR(t)} = \sigma^2\boldsymbol{(t)}\, \boldsymbol{d}\mathrm{t}$

Let dt = 0, we obtain the PDE $\boldsymbol{f_t(t,r) + (a(t) - b(t)r)f_r(t,r) + \frac{1}{2}\sigma^2(t)f_{rr}(t,r) = rf(t,r)}$ (6.5.6)

Example 6.5.1 (Hull-White interest rote model)

To find the $f(t,r)$ that satisify (6.5.6)

We initially guess and subsequently verify that the solution has the form

$$\mathbf{f}(t,r) = e^{-rC(t,T)-A(t.T)}$$

(for some nonrandom functions C(t,T) and A(t,T) to be determined of t ∈ [0, T])

In this case, the yield $Y(t,T) = \frac{1}{T-t}\log f(t,r) = \frac{1}{T-t}(rC(t,T) + A(t,T))$

(an affine function of r (i.e., a number times r plus another number)

Furthermore,

$f_t(t,r) = \left(-rC'^{(t,T)} - A'(t,T)\right) f\ (t,r)$ (Both C and A are differentiated with respect to t)

$f_r(t,r) = -C(t,T)f(t,r)$

$f_r(t,r) = C^2(t,T)f(t,r)$

Example 6.5.1 (Hull-White interest rote model)

Substitution into the PDE (6.5.6) gives

$$\left(-r\boldsymbol{C}'(t,T)-\boldsymbol{A}'(t,T)\right)f\ (t,r)+(a(t)-b(t)r)*-C(t,T)f(t,r)+\frac{1}{2}\sigma^2C^2(t,T)f(t,r)-rf=0$$

$$\Rightarrow [(-C'(t,T)-b(t)C(t,T)-1)r\text{-}A'(t,T)-a(t)C(t,T)+\frac{1}{2}\sigma^2C^2(t,T)]\ f(t,r)=0 \quad (6.5.7)$$

Because this equation must hold for all r,that

$$\text{-}C'(t,T)-b(t)C(t,T)-1)=0\Rightarrow C'(t,T)=b(t)C(t,T)-1) \quad (6.5.8)$$

$$-A'(t,T)-a(t)C(t,T)+\frac{1}{2}\sigma^2C^2(t,T)=0\Rightarrow A'(t,T)=-a(t)C(t,T)+\frac{1}{2}\sigma^2C^2(t,T) \quad (6.5.9)$$

And it must also meet the terminal conditions of 6.5.5 , that $\mathbf{C}(\mathbf{T},\mathbf{T})=\mathbf{A}(\mathbf{T},\mathbf{T})=0$

Therefore, we can obtain

$$C(t,T)=\int_t^{\mathrm{T}}e^{-\int_t^s b(u)du}ds \quad (6.5.10)$$

$$A(t,T)=\int_t^T(a(s)C(s,T)-\frac{1}{2}\sigma^2C^2(t,T))ds \quad (6.5.11)$$

在$\boldsymbol{C}(\boldsymbol{t},\boldsymbol{T})$，我們知$C'(t,T) - b(t)C(t,T) = -1$ 並將整個式子同乘積分因子$u(t) = e^{\int^{t} b(u)du}$

就可以得到$u(t)C'(t,T) - u(t)b(t)C(t,T) = -\mathrm{u(t)}$

其中$\frac{d}{dt}\mathrm{u(t)C(t,T)} = u(t)C'(t,T) + u'(t)C(t,T) \Rightarrow -u'(t) = -b(t)u(t) \Rightarrow u(t)C'(t,T) + -b(t)u(t)C(t,T)$

所以$\frac{d}{dt}\mathrm{u(t)C(t,T)} = -\mathrm{u(t)}$

因此當對$\frac{d}{dt}\mathrm{u(t)C(t,T)}$積分時$\Rightarrow \int_t^T u(t)C'(t,T) = u(T)C(T,T) - u(t)C(t,T) \Rightarrow$ 因為終端條件C(T,T)=0

$\Rightarrow -u(t)C(t,T) = -\int_t^T u(s)ds \Rightarrow C(t,T) = \frac{\int_{\mathrm{t}}^{\mathrm{T}} -\mathrm{u(s)ds}}{\mathrm{u(t)}} \Rightarrow$ 其中$\frac{u(s)}{u(t)} = e^{\int_t^s b(u)du}$

最後可以得$\boldsymbol{C}(\boldsymbol{t},\boldsymbol{T}) = \int_{\boldsymbol{t}}^{\mathrm{T}} \boldsymbol{e}^{-\int_t^s \boldsymbol{b}(\boldsymbol{u})\boldsymbol{du}}\boldsymbol{ds}$ (6.5.10)

而對A$(\boldsymbol{t},\boldsymbol{T})$來說直接對$\boldsymbol{A}'(\boldsymbol{t},\boldsymbol{T})$積分$\Rightarrow \int_t^T A'(t,T) = A(T,T) - A(t,T) = \int_t^T -a(s)C(s,T) + \frac{1}{2}\sigma^2C^2(s,T)ds$

⇒因為終端條件A(T,T) = 0 ，所以$A(t,T) = \int_t^T (a(s)C(s,T) - \frac{1}{2}\sigma^2C^2(s,T)ds$

Example 6.5.3 (Option on a bond).

Consider the general short-rate model (6.5.1) $\boldsymbol{dR(t) = \beta(t, R(t))\, dt + \gamma(t, R(t))\, d\widetilde{W}(t)}$

Let $0 \le t \le T_1 \le T_2$. In this example, T_2 = zero-bond maturity , T_1 = European call option maturity

We wish to determine the value of this call at time t

Suppose we have solved for the function f(t,r) satisfying the partial differential equation (6.5.4) together

with the terminal condition (6.5.5)

According to the risk-neutral pricing formula (5.2.31) and the Markov property,

the value of the call at time t is

$$\boldsymbol{c\big(t, R(t)\big) = \widetilde{E}\left[e^{-\int_t^{T_1} R(s)ds}\big(f\big(T_1, R(T_1)\big) - K\big)^+ \big| F(t)\right] = \frac{1}{D(t)}\widetilde{E}\left[\big(D(T_1)\big(f\big(T_1, R(T_1)\big) - K\big)^+ \big| F(t)\right]}$$

discounted call price $\boldsymbol{\mathrm{D(t)}c\big(t, R(t)\big) = \widetilde{E}\left[D(T_1)\big(f\big(T_1, R(T_1)\big) - K\big)^+ \big| F(t)\right], 0 \le t \le T_1}$

Example 6.5.3 (Option on a bond).

The discounted call price is a martingale ,by discounted Feynman-Kac Theorm

The diflerential of the discounted call price is

$$\mathbf{d}\left(\mathbf{D(t)}c\big(t,R(t)\big)\right) = c\big(t,R(t)\big)dD(t) + D(t)dc\big(t,R(t)\big)$$

$$= D\left[-Rcdt + c_t dt + c_r dR + \frac{1}{2}C_{rr}dRdR\right] = D\left[-Rc + c_t + \beta c_r + \frac{1}{2}\gamma^2 c_{rr}\right]dt + D\gamma c_r d\widetilde{W}$$

Because martingale that dt=0

$$c_t(t,r) + \beta(t,r)c_r(t,r) + \frac{1}{2}\gamma^2(t,r)c_{rr}(t,r) = rc(t,r)$$

The terminal condition for c(t,r) is

$c(T_1,r) = (f(T_1,r) - K)^+$ **for all r**