5.2.3
Value of Portfolio Process Under the Risk-Neutral Measure

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1. Consider begins with initial capital $X(0)$
2. At each time t, $0 \leq t \leq T$, holds $\Delta(t)$ shares of stock, and investing or borrowing at the interest rate $R(t)$ as necessary to finance this.

Where $\alpha(t), \sigma(t), R(t)$ are random, the differential of the portfolio value becomes：

$$dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt$$

$$= \Delta(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) + R(t)(X(t) - \Delta(t)S(t))dt \text{ (移項)}$$

$$= R(t)X(t)dt + \Delta(t)(\alpha(t) - R(t))S(t)dt + \Delta(t)\sigma(t)S(t)dW(t)$$

$$= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)] \qquad (5.2.25)$$

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \qquad \Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

Now derive the differential of the discounted portfolio value：

Itô's product rule： $d(D(t)X(t)) = dD(t)X(t)+D(t)dX(t) + dD(t)dX(t)$

imply (5.2.18) $dD(t) = -R(t)D(t)dt$

$d(D(t)X(t)) = -R(t)D(t)X(t)dt + D(t)[R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt+dW(t)]]$

$= D(t)\Delta(t)\sigma(t)S(t)[\Theta(t)dt+dW(t)]$

imply (5.2.20) $d(D(t)S(t)) = \sigma(t)D(t)S(t)[\Theta(t)dt+dW(t)]$

$d(D(t)X(t)) = \Delta(t)\, d(D(t)S(t))$ (5.2.26)

Under the new measure $\tilde{P}$ defined by $\Theta(t)dt$， $d(D(t)S(t)) = \sigma(t)D(t)S(t)\, d\widetilde{W}(t)$

$d(D(t)X(t)) = \Delta(t)\,\sigma(t)D(t)S(t)\, d\widetilde{W}(t)$ (5.2.27)

d(D(t)X(t)) is a martingale under zero drift

5.2.4
Pricing Under the Risk-Neutral Measure

Initial capital X (0) and portfolio process $\Delta(t)$ an agent would need in order to hedge a short position in the call (i.e., In order to have $X(T) = (S(T) - K)^+$ almost surely

Let V(T) be an F(T)-measurable random variable. This represents the payoff at time T of a derivative security

$X(T) = V(T)$ almost surely. (5.2.28) Then $D(T)X(T) = D(T)V(T)$ also holds

Since $D(t)X(t)$ is a martingale under the probability measure $\widetilde{P}$, it follows that

$$D(t)X(t)= \widetilde{E}[D(T)X(T) \mid F(t)] = \widetilde{E}[D(T)V(T) \mid F(t)] \qquad (5.2.29)$$

That $D(t)V(t)= \widetilde{E}[\, D(T)V(T) \mid F(t)\,], 0 \le t \le T$ (5.2.30)

Since D(t), which is F(t)-measurable, is a known and deterministic value at time t

Dividing both sides by D(t), we obtain： $V(t)= \widetilde{E}\left[e^{-\int_t^T R(u)du} V(T) \,\middle|\, F(t) \right], 0 \le t \le T$ (5.2.31)

5.2.5
Deriving the Black-Scholes-Merton Formula

To obtain the Black-Scholes-Merton price of a European call, we assume a constant volatility σ, constant interest rate r, and take the derivative security payoff to be $V(T) = (S(T) — K)^+$.

The right-hand side of (5.2.31) becomes $\widetilde{E}\left[e^{-r(T-t)}(S(T) — K)^+ \mid F(t) \right]$

Because geometric Brownian motion is a Markov process, this expression depends on the stock price S(t) and of course on the time t at which the conditional expectation is computed, but not on the stock price prior to time t

Therefore, define a function $c(t,S(t)) = \widetilde{E}\left[e^{-r(T-t)}(S(T) — K)^+ \mid F(t) \right]$

$$= \widetilde{E}\left[e^{-r(T-t)}(S(T) — K)^+ \mid S(t) \right]$$

With constant σ and r, equation (5.2.24) becomes $S(t) = S(0)e^{\{\sigma\widetilde{W}(t)+\left(r-\frac{1}{2}\sigma^2\right)t\}}$

$\rightarrow S(T) = S(t)e^{\{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+\left(r-\frac{1}{2}\sigma^2\right)\tau\}}$ Now, let Y= $-\frac{\widetilde{W}(T)-\widetilde{W}(t)}{\sqrt{T-t}}$

$\rightarrow S(T) = S(t)e^{\{-\sigma\sqrt{\tau}Y)+\left(r-\frac{1}{2}\sigma^2\right)\tau\}}$

Using the Independence Lemma, the random variable S(T) can be decomposed into the product of an F(t)-measurable random variable S(t) and another random variable $e^{\{-\sigma\sqrt{\tau}Y)+\left(r-\frac{1}{2}\sigma^2\right)\tau\}}$, where $e^{\{-\sigma\sqrt{\tau}Y)+\left(r-\frac{1}{2}\sigma^2\right)\tau\}}$ is independent of F(t)

Therefore, (5.2.32) holds with

$$c(t,x) = \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}Y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+ e^{-\frac{1}{2}y^2}\,dy.$$

$(xe^{\{-\sigma\sqrt{\tau}y)+\left(r-\frac{1}{2}\sigma^2\right)\tau\}}— K)+$, is positive if and only if $y < d_-(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}[log\frac{x}{K} + (r - \frac{1}{2}\sigma^2)\tau]$ (5.2.23)

Therefore,

$$
\begin{aligned}
c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}\,y+\left(r-\frac{1}{2}\sigma^2\right)\tau\right\}-K\right)e^{-\frac{1}{2}y^2}\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} x\exp\left\{-\frac{y^2}{2}-\sigma\sqrt{\tau}\,y-\frac{\sigma^2\tau}{2}\right\}dy \quad \leftarrow -r\tau-\sigma\sqrt{\tau}y+(r-\tfrac{1}{2}\sigma^2)\tau\text{-}\tfrac{1}{2}y^2 \\
&\quad -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}Ke^{-\frac{1}{2}y^2}\,dy \quad \leftarrow y\sim N(0,1) \\
&= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} \exp\left\{-\frac{1}{2}\left(y+\sigma\sqrt{\tau}\right)^2\right\}dy - e^{-r\tau}KN\left(d_-(\tau,x)\right) \quad \leftarrow (a+b)^2 = a^2+2ab+b^2 \\
&= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)+\sigma\sqrt{\tau}} \exp\left\{-\frac{z^2}{2}\right\}dz - e^{-r\tau}KN\left(d_-(\tau,x)\right) \quad \leftarrow z=\left(y+\sigma\sqrt{\tau}\right) \\
&\qquad \leftarrow \text{上界：}\ d_-(\tau,x)+\sigma\sqrt{\tau} \\
&= xN\left(d_+(\tau,x)\right) - e^{-r\tau}KN\left(d_-(\tau,x)\right),
\end{aligned}
$$

where

$$
d_+(\tau,x) = d_-(\tau,x)+\sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K}+\left(r+\frac{1}{2}\sigma^2\right)\tau\right]. \qquad (5.2.34)
$$

For future reference, we introduce the notation

$$\mathrm{BSM}(\tau, x; K, r, \sigma) = \widetilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left\{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right]$$

where Y is a standard normal random variable under $\widetilde{\mathbb{P}}$. We have just shown that

$$\mathrm{BSM}(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - e^{-r\tau}KN(d_-(\tau, x)). \qquad (5.2.36)$$

5.3.1
Martingale Representation with One Brownian Motion

The risk-neutral pricing formula (5.2.31) assumes that with the correct initial capital, there exists a portfolio process $\Delta(t), 0 \le t \le T$, such that the portfolio value equals $V(T)$ almost surely at time T.

Under this assumption, the correct initial capital is given by setting $t=0$ in equation (5.2.31).

$V(t) = \widetilde{E}[D(T)V(T)]$ and the value of the hedging portfolio at every time t, $0 \le t \le T$, to be $V(t)$ given by (5.2.31).

We verify that in the model with a single stock driven by one Brownian motion, the assumption underlying the risk-neutral pricing formula (5.2.31)—namely, the existence of a hedging portfolio—is justified by the following theorem.

Theorem 5.3.1 (Martingale representation, one dimension).

Let W(t) be a Brownian motion defined on a probability space$(\Omega, \mathcal{F}, \mathrm{P})$

Let F(t) be the filtration generated by W(t), and suppose that M(t) is a martingale with respect to this filtration F(t)

Theorem 5.3.1 states that if the filtration F(t) is generated by a Brownian motion W(t), then any martingale M(t) adapted to this filtration F(t) can be expressed as:

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u), 0 \leq t \leq T$$

← cannot have jumps because Itô integrals are continuous

can be expressed as an initial value plus an Itô integral with respect to Brownian motion

Theorem 5.3.1 imposes stricter conditions than Theorem 5.2.3 (Girsanov's Theorem, one-dimensional case)

Corollary 5.3.2

We now want to show that, after a change of probability measure $\tilde{P}$, the new martingales can still be represented in the form of an Itô integral

Therefore, we apply Theorem 5.2.3 (Girsanov's Theorem, one-dimensional case) to perform the change of measure and carry out the verification

Let W(t), 0 ≤ t ≤ T, be a Brownian motion on a probability space (Ω, F, P), and let F(t), 0 ≤ t ≤ T, be a filtration for this Brownian motion. Let$\Theta(t), 0 \leq t \leq T$, be an adapted process.

Define $Z(t) = e^{\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}}$, $\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$ ← Z(t) as a weight

← $\widetilde{W}$(t) under the measure $\tilde{P}$

and assume that $\tilde{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty, Set\ Z = Z(T), then\ EZ = 1$

Now, let M(t), 0 ≤ t ≤ T, be a martingale under $\tilde{P}$. Then there is an adapted process $\Gamma(u)$, 0 ≤ u ≤ T, such that $\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{W}(u),\ 0 \leq t \leq T.$ (5.3.2)

5.3.2
Hedging with One Stock

We begin with the model of Subsection 5.2.2, which has the stock price process

(5.2.15) $dS(t) = \alpha(t)S(t)dt+\sigma(t)S(t)dW(t)$ and an interest rate process R(t) that

generates the discount process (5.2.17) $D(t) = e^{\int_0^t R(s)ds}$

Recall the assumption that, for [0,T], the volatility σ(t) is almost surely not zero.

We make the additional assumption that the filtration F(t), $0 \leq t \leq T$, is generated by the Brownian motion W(t), $0 \leq t \leq T$

Let V(T) be an F(T)-measurable random variable and, for $0 \leq t \leq T$, define V(t) by the risk-neutral pricing formula (5.2.31). Then, according to (5.2.30), $D(t)V(t)= \widetilde{E}[\, D(T)V(T) \mid F(t)\,]$

This is a $\tilde{P}$-martingale; indeed, iterated conditioning implies that, for 0 < s < t

$$\tilde{E}[\, D(T)V(T) \mid F(t)\,] = \tilde{E}\big[\tilde{E}[\, D(T)V(T) \mid F(t)\,]\big|F(s)\big] = \tilde{E}[\, D(T)V(T) \mid F(s)\,] = D(s)V(s) \quad (5.3.3)$$

According Theorem5.3.1 & Corollary 5.3.2 know that D(t)V (t) has a representation as

$D(t)V(t) = V(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), 0 \le t \le T$ (5.3.4) , (recall that D(0)V (0) = V (0))

On the other hand, for any portfolio process Δ(t), the differential of the discounted portfolio value is given by (5.2.27), and hence $D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)\, d\widetilde{W}(u),\ 0 \le t \le T.$ (5.3.5)

In order to have X (t) = V(t) for all t, we should choose X(0)=V(0) (5.3.6)

and choose Δ(t) to satisfy $\Delta(t)\sigma(t)D(t)S(t) = \tilde{\Gamma}(u)$, $0 \le t \le T$ (5.3.7)

which is equivalent to $\Delta(t) = \dfrac{\tilde{\Gamma}(u)}{\sigma(t)D(t)S(t)}$, $0 \le t \le T$ (5.3.8)

A hedge for a short position in the derivative security with payoff V (T) at time T.

Two key assumptions that make the hedge possible.

The first is that the volatility o(t) is not zero, so equation (5.3.7) can be solved

The other key assumption is that F(t) is generated by the underlying Brownian

Under these two assumptions, every F(T)-measurable derivative security can be hedged

Such a model is said to be complete.