

Black-Scholes-Merton Equation

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- We derive the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as a geometric Brownian motion.
- Determine the initial capital required to perfectly hedge a short position in the option.

4.5.1 Evolution of Portfolio Value

- A portfolio valued $X(t)$ at each time t .
- This portfolio invests in a money market account paying a constant rate of interest r and in a stock modeled by the geometric Brownian motion

$$\text{Stock price : } dS(t) = \alpha S(t) dt + \sigma S(t) dW(t) \quad (4.5.1)$$

4.5.1 Evolution of Portfolio Value

- Suppose at each time t , holds $\Delta(t)$ shares of stock.
- $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$, $t > 0$.
- Invested in the money market account is $X(t) - \Delta(t)S(t)$.

$$\begin{aligned}dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t)) dt \\ &= \Delta(t)(\alpha S(t) dt + \sigma S(t) dW(t)) + r(X(t) - \Delta(t)S(t)) dt \\ &= r(X(t) dt + \Delta(t)(\alpha - r)S(t) dt + \Delta(t)\sigma S(t) dW(t))\end{aligned}\tag{4.5.2}$$

4.5.1 Evolution of Portfolio Value

The discrete-time analogue of equation (4.5.2) appears in Chapter 1 of Volume I as (1.2.12):

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n).$$

We may rearrange terms in this equation to obtain

$$X_{n+1} - X_n = \Delta_n (S_{n+1} - S_n) + r(X_n - \Delta_n S_n). \quad (4.5.3)$$

which is analogous to the first line of (4.5.2), except in (4.5.3) time steps forward one unit at a time, whereas in (4.5.2) time moves forward continuously.

4.5.1 Evolution of Portfolio Value

According to the Ito-Doebelin formula with $f(t, x) = e^{-rt} x$

Consider the discounted stock price $e^{-rt} S(t)$

$$\begin{aligned}d(e^{-rt} S(t)) &= df(t, S(t)) = f_t(t, S(t)) dt + f_x(t, S(t)) dS(t) + \frac{1}{2} f_{xx}(t, S(t)) dS(t)dS(t) \\ &= -re^{-rt} S(t) dt + e^{-rt} dS(t) + 0 \\ &= (a - r)e^{-rt} S(t) dt + \sigma e^{-rt} S(t) dW(t)\end{aligned}\tag{4.5.4}$$

Consider the discounted portfolio value $e^{-rt} X(t)$.

$$\begin{aligned}d(e^{-rt} X(t)) &= df(t, X(t)) = f_t(t, X(t)) dt + f_x(t, X(t)) dX(t) + \frac{1}{2} f_{xx}(t, X(t)) dX(t)dX(t) \\ &= -re^{-rt} X(t) dt + e^{-rt} dX(t) + 0 \\ &= -re^{-rt} X(t)dt + e^{-rt}(r(X(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t))) \\ &= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) = \Delta(t) d(e^{-rt} S(t))\end{aligned}\tag{4.5.5}$$

4.5.2 Evolution of Option Value

- A European call option that pays $(S(T) - K)^+$ at time T and the strike price K is some nonnegative constant.
- Black, Scholes, and Merton argued that the value of call at any time should depend on the time (more precisely, on the time to expiration) 、 stock price at that time 、 the model parameters σ & u and the contractual strike price K .
- Only two of these quantities, time(t) and stock price $S(t)$ are variable.

4.5.2 Evolution of Option Value

- Let $c(t, x)$ denote the value of the call at time t if the stock price at that time is $S(t) = x$.
- There is not random about the function $c(t, x)$. However, the value of the option is random; it is the stochastic process $c(t, S(t))$ obtained by replacing the dummy variable x by the random stock price $S(t)$ in this function.
- Our goal is to determine the function $c(t, x)$, so we at least have a formula for the future option values in terms of the future stock prices.

4.5.2 Evolution of Option Value

According to the Ito-Doebelin formula to computing the differential of $c(t, S(t))$

$$\begin{aligned}dc(t, S(t)) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2} c_{xx}(t, S(t))dS(t)dS(t) \\&= c_t(t, S(t))dt + c_x(t, S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2} c_{xx}(t, S(t))\sigma^2 S^2(t)dt \\&= [c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t))] dt \\&\quad + \sigma S(t) c_x(t, S(t))dW(t)\end{aligned}\tag{4.5.6}$$

$$dS(t) * dS(t) = \alpha S(t) dt + \sigma S(t) dW(t) * \alpha S(t) dt + \sigma S(t) dW(t) = \sigma^2 S^2(t) dt$$

4.5.2 Evolution of Option Value

According to the Ito-Doebelin formula , let $f(t, x) = e^{-rt} x$ to compute the differential of the discounted option price $e^{-rt}c(t, S(t))$.

$$\begin{aligned}d(e^{-rt} c(t, S(t))) &= df(t, c(t, S(t))) \\&= f_t(t, c(t, S(t)))dt + f_x(t, c(t, S(t))) dc(t, S(t)) \\&\quad + \frac{1}{2} f_{xx}(t, c(t, S(t))) dc(t, S(t)) dc(t, S(t)) \\&= -re^{-rt}(c(t, S(t))) dt + e^{-rt} dc(t, S(t)) + 0 \\&= -re^{-rt}(c(t, S(t))) dt + e^{-rt} \{[c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \\&\quad + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t))] dt + \sigma S(t) c_x(t, S(t))dW(t)\} \\&= e^{-rt} [-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \\&\quad + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t))] dt + e^{-rt} \sigma S(t) c_x(t, S(t))dW(t)\end{aligned}\tag{4.5.7}$$

4.5.3 Equating the Evolutions

- A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account
- The portfolio value $X(t)$ at each time $t \in [0, T]$ agrees with $c(t, S(t))$. This happens if and only if $e^{-rt} X(t) = e^{-rt} c(t, S(t))$ for all t .
- One way to ensure this equality is to make sure that

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t))) \text{ for all } t \in [0, T) \quad (4.5.8)$$

and $X(0) = c(0, S(0))$. Integration of (4.5.8) from 0 to t then yields

$$e^{-rt}X(t) - X(0) = e^{-rt}c(t, S(t)) - c(0, S(0)) \text{ for all } t \in [0, T) \quad (4.5.9)$$

4.5.3 Equating the Evolutions

- Comparing (4.5.5) and (4.5.7), we see that (4.5.8) holds if and only if

$$\begin{aligned} & \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\ &= [-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t))] dt \\ & \quad + \sigma S(t)c_x(t, S(t))dW(t) \end{aligned} \tag{4.5.10}$$

We examine what is required in order for (4.5.10) to hold.

We first equate the $dW(t)$ terms in (4.5.10), which gives

$$\Delta(t) = c_x(t, S(t)) \quad \text{for all } t \in [0, T) \tag{4.5.11}$$

4.5.3 Equating the Evolutions

- This quantity, $c_x(t, S(t))$, is called the *delta* of the option.
We next equate the dt terms in (4.5.10), using (4.5.11), to obtain

$$\begin{aligned} & (\alpha - r)S(t) c_x(t, S(t)) \\ &= -rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t)) \end{aligned}$$

for all $t \in [0, T)$ (4.5.12)

The term $\alpha S(t)c_x(t, S(t))$ appears on both sides of (4.5.12), and after canceling it, we obtain

$$\begin{aligned} rc(t, S(t)) &= c_t(t, S(t)) + rS(t)c_x(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) c_{xx}(t, S(t)) \end{aligned}$$

for all $t \in [0, T)$ (4.5.13)

4.5.3 Equating the Evolutions

- In conclusion, we should seek a continuous function $c(t, x)$ that is a solution to the Black-Scholes-Merton partial differential equation

$$c_t(t, x) + rx c_x(t, x) + \frac{1}{2} \sigma^2 x^2 c_{xx}(t, x) = rc(t, x) \quad \text{for all } t \in [0, T), x \geq 0, \quad (4.5.14)$$

and that satisfies the *terminal condition*

$$c(T, x) = (X - K)^+ \quad (4.5.15)$$

4.5.3 Equating the Evolutions

- Initial capital $X(0) = c(0, S(0))$ and uses the hedge $\Delta(t) = cx(t, S(t))$, then (4.5.10) will hold for all $t \in [0, T)$.
- Because $\Delta(t) = cx(t, S(t))$, the $dW(t)$ terms on the left and right sides of (4.5.10) agree, and the dt terms agree because (4.5.14) guarantees (4.5.13).
- Equality in (4.5.10) gives us (4.5.9). Canceling $X(0) = c(0, S(0))$ and e^{-rt} in this equation, we see that $X(t) = c(t, S(t))$ for all $t \in [0, T)$.

4.5.3 Equating the Evolutions

- Taking the limit as t to T and using the fact that both $X(t)$ and $c(t, S(t))$ are continuous, we conclude that
$$X(T) = c(T, S(T)) = (S(T) - K)^+.$$
- This means that the short position has been successfully hedged.
No matter which of its possible paths the stock price follows, when the option expires, the agent hedging the short position has a portfolio whose value agrees with the option payoff.