Stochastic Calculus For Finance - volume 2

- Section 8.3 Perpetual American Put

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Definition 8.3.1.

Let $\mathscr{T}$ be the set of all stopping times. The price of the perpetual American put is defined to be

$$v_*(x) = \max_{\tau \in \mathscr{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right]$$

where x = S(0) in (8.3.2) is the initial stock price. In the event that $\tau = \infty$, we interpret $e^{-r\tau}\big(K - S(\tau)\big)$ to be zero.

- The holder of a perpetual American put may choose any exercise time τ, without looking into the future.
- Mathematically, this "no look-ahead" restriction means

→ τ must be a stopping time.

- If the option is exercised at time τ, the payoff is K - S(τ).
- If the option is never exercised ($\tau = \infty$), the payoff is zero.

Risk-Neutral Valuation

- The t=0 price of the option is the risk-neutral expected payoff, discounted back to time 0:

$$v_*(x) = \max_{\tau \in \mathcal{T}} \widetilde{\mathbb{E}}\left[e^{-r\tau}\big(K - S(\tau)\big)\right]$$

- The optimal exercise strategy is the one that maximizes this discounted expected payoff.

Perpetual Feature and Exercise Timing

- A perpetual American put has no expiration date.
- Therefore, every date is identical
- the remaining time to maturity is always infinite.

As a result, the optimal exercise decision:

- does not depend on time,
- depends only on the current stock price S(t).

Threshold-Type Optimal Policy

- It is natural to expect the holder to exercise only when the stock price is sufficiently low.
- Hence, the optimal exercise rule takes a threshold form:

Exercise the put as soon as } $S(t) \leq L_{*}$.

Theorem 8.3.2

(Laplace transform for first passage time of drifted Brownian motion).

Let $\widetilde{W}(t)$be a Brownian motion under a probability measure $\widetilde{\mathbb{P}}$, let μ be a real number, and let m be a positive number.

Define $X(t) = \mu t + \widetilde{W}(t),$ and set $\tau_m = \min\{t \geq 0 : X(t) = m\},$ so that τ_m is the stopping time

If $X(t)$ never reaches the level m, then we interpret τ_m to be ∞.

Then

$$\widetilde{\mathbb{E}}\left[e^{-\lambda\tau_m}\right] = e^{-m\left(-\mu+\sqrt{\mu^2+2\lambda}\right)} \quad \text{for all } \lambda > 0, \qquad (8.3.3)$$

where we interpret $e^{-\lambda\tau_m}$ to be zero if $\tau_m = \infty$.

$$= -\frac{1}{2}\sigma^2 t$$

$$\sigma X(t) - \lambda t = \sigma(\mu t + \widetilde{W}(t)) - \lambda t = (\sigma\mu - \lambda)t + \widetilde{W}(t)$$

$$\Rightarrow (\sigma\mu - \lambda)t = -\frac{1}{2}\sigma^2 t \Rightarrow \lambda = \sigma\mu + \frac{1}{2}\sigma^2 t$$

proof:

Define $\sigma = -\mu + \sqrt{\mu^2 + 2\lambda}$ so that $\sigma > 0$ and

$$\sigma\mu + \frac{1}{2}\sigma^2 = -\mu^2 + \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)^2$$

$$= -\mu^2 + \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\mu^2 - \mu\sqrt{\mu^2 + 2\lambda} + \frac{1}{2}\mu^2 + \lambda = \lambda$$

Then

$$e^{\sigma X(t) - \lambda t} = e^{\sigma\mu t + \sigma \widetilde{W}(t) - \sigma\mu t - \frac{1}{2}\sigma^2 t} = e^{\sigma \widetilde{W}(t) - \frac{1}{2}\sigma^2 t}$$, which is a martingale under $\widetilde{\mathbb{P}}$

According to Theorem 8.2.4 (optional sampling),the stopped martingale

$M(t) = e^{\sigma \widetilde{W}(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)}$ is also a martingale.

Therefore, for each positive integer n,

$$1 = M(0) = \widetilde{\mathbb{E}}[M(n)] = \widetilde{\mathbb{E}}\left[e^{\sigma X(n \wedge \tau_m) - \lambda(n \wedge \tau_m)}\right] = \widetilde{\mathbb{E}}\left[e^{\sigma m - \lambda \tau_m} \mathbf{1}_{\{\tau_m \le n\}}\right] + \widetilde{\mathbb{E}}\left[e^{\sigma X(n) - \lambda n} \mathbf{1}_{\{\tau_m > n\}}\right]. \quad (8.3.4)$$

① ②

proof:

① $\widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le n\}}\right]$

The nonnegative random variables $e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le n\}}$ increase with n

$0\le e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le 1\}}\le e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le 2\}}\le\cdots$almost surely. and

$\lim\limits_{n\to\infty} e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le n\}}=e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m<\infty\}}$ {almost surely}.

The Monotone Convergence Theorem, implies $\lim\limits_{n\to\infty}\widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m\le n\}}\right]=\widetilde{\mathbb{E}}\left[e^{\sigma m-\lambda\tau_m}\mathbf{1}_{\{\tau_m<\infty\}}\right]$.

② $\widetilde{\mathbb{E}}\left[e^{\sigma X(n)-\lambda n}\mathbf{1}_{\{\tau_m>n\}}\right]$

the random variable $e^{\sigma X(n)-\lambda n}\mathbf{1}_{\{\tau_m>n\}}$ satisfies $0\le e^{\sigma X(n)-\lambda n}\mathbf{1}_{\{\tau_m>n\}}\le e^{\sigma m-\lambda n}\le e^{\sigma m}$ almost surely

because $X(n)\le m$ for $n<\tau_m$ and σ is positive.

Because λ is positive, we have

$\lim\limits_{n\to\infty} e^{\sigma X(n)-\lambda n}\mathbf{1}_{\{\tau_m>n\}}\le\lim\limits_{n\to\infty} e^{\sigma m-\lambda n}=0.$

According to the Dominated Convergence Theorem $\lim\limits_{n\to\infty}\widetilde{\mathbb{E}}\left[e^{\sigma X(n)-\lambda n}\mathbf{1}_{\{\tau_m>n\}}\right]=0$ (8.3.6)

Taking the limit in (8.3.4) and using (8.3.5) and (8.3.6), we obtain

$$1 = \widetilde{\mathbb{E}}\left[e^{\sigma m - \lambda \tau_m} \mathbf{1}_{\{\tau_m < \infty\}}\right].$$

Or, equivalently,

$$[\widetilde{\mathbb{E}}\left[e^{-\lambda \tau_m} \mathbf{1}_{\{\tau_m < \infty\}}\right] = e^{-\sigma m} = e^{-m\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)} \qquad \text{for all } \lambda > 0. \quad (8.3.7)$$

$$1 = \tilde{E}\left[e^{\sigma m - \lambda \tau m} \mathbf{1}_{\{\tau m < \infty\}}\right] + 0 \qquad ① \quad ②$$

$$= e^{\sigma m} \tilde{E}\left[e^{-\tau m} \mathbf{1}_{\{\tau m < \infty\}}\right]$$

$$\Rightarrow \tilde{E}\left[e^{-\tau m} \mathbf{1}_{\{\tau m < \infty\}}\right] = e^{-\sigma m} = e^{-m\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)}$$

Remark 8.3.3.

We used the strict positivity of ν to derive (8.3.7), but now that we have it, we can take the limit as $\lambda \downarrow 0$. The random variables $e^{-\lambda\tau_m}\mathbf{1}_{\{\tau_m<\infty\}}$ are nonnegative and increase to $\mathbf{1}_{\{\tau_m<\infty\}}$ as $\lambda \downarrow 0$, and the Monotone Convergence Theorem allows us to conclude that

$$\widetilde{\mathbb{P}}\{\tau_m < \infty\} = \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tau_m<\infty\}}\right] = \lim_{\lambda\downarrow 0} e^{-m\left(-\mu+\sqrt{\mu^2+2\lambda}\right)} = e^{m\mu-m|\mu|}.$$

If $\mu \geq 0$, the drift in $X(t)$ is zero or upward, toward level m, and $\widetilde{\mathbb{P}}\{\tau_m < \infty\} = 1$; the level $X(t)$ is reached with probability one.

On the other hand, if $\mu < 0$, the drift in $X(t)$ is downward, away from level m, and

$$\widetilde{\mathbb{P}}\{\tau_m < \infty\} = e^{-2m|\mu|} < 1;$$

there is a positive probability of never reaching m.

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}. \quad \textbf{(8.3.8)}$$

Suppose the owner of the perpetual American put sets a positive level $L < K$ and resolves to exercise the put the first time the stock price falls to L.

① If the initial stock price is at or below L , she exercises immediately (at time zero).

The value of the put in this case is $v_L(S(0)) = K - S(0)$.

② If the initial stock price is above L , she exercises at the stopping time

$\tau_L = \min\{t \geq 0 : S(t) = L\}$, (8.3.9)

where τ_L is set equal to ∞ if the stock price never reaches the level L.

At the time of exercise, the put pays $K - S(\tau_L) = K - L$.

Discounting this back to time zero and taking the risk-neutral expected value, we compute the value of the put under this exercise strategy to be

$v_L(S(0)) = (K - L)\,\widetilde{\mathbb{E}}\left[e^{-r\tau_L}\right]$ for all $S(0) \geq L$ (8.3.10)

On those paths where $\tau_L = \infty$, we interpret $e^{-r\tau_L}$ to be zero.(r is positive.)

Although not explicitly indicated by the notation, the distribution of τ_L depends on the initial stock price $S(0)$, so the right-hand side of (8.3.10) is a function of $S(0)$.

Lemma 8.3.4

The function $v_L(x)$ is given by the formula

$$v_L(x) = \begin{cases} K - x, & 0 \le x \le L, \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L. \end{cases}$$

proof:

We only need to establish the second line.

If $x = L$, then $\tau_L = 0$ and(8.3.10) implies $v_L(x) = K - L$

Theorem 8.3.2

$$X(t) = \mu t + \widetilde{W}(t),$$

We consider the case $S(0) = x > L$.

The stopping time τ_L is the first time reaches the level L.

$$S(t) = x \exp\left\{ \sigma \widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right) t \right\}$$

$S(t) = L$

$\Rightarrow x \exp\{\sigma \widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\} = L$

$\Rightarrow \exp\{\sigma \widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\} = \frac{L}{x}$

$\Rightarrow \sigma \widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t = \log(\frac{L}{x})$

$\Rightarrow -\widetilde{W}(t) - \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)t = \frac{1}{\sigma}\log(\frac{x}{L})$

But $S(t) = L$ if and only if

$$-\widetilde{W}(t) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) t = \frac{1}{\sigma} \log \frac{x}{L}.$$

We now apply Theorem8.3.2 with $X(t)$ in that theorem replaced by

$$-\widetilde{W}(t) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) t$$

(the processes $\widetilde{W}(t)$ and $-\widetilde{W}(t)$ are both Brownian motions under $\widetilde{\mathbb{P}}$,with λ replaced by r, with μ replaced by $-\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$, and with m replaced by $\frac{1}{\sigma} \log \frac{x}{L}$, which is positive.

Theorem 8.3.2

$$\widetilde{\mathbb{E}} e^{-\lambda\tau_m} = e^{-m\left(-\mu+\sqrt{\mu^2+2\lambda}\right)}$$

With these replacements, τ_m in Theorem 8.3.2 is τ_L, and

$$\mu^2 + 2\lambda = \frac{1}{\sigma^2}\left(r^2 - r\sigma^2 + \tfrac{1}{4}\sigma^4\right) + 2r = \frac{1}{\sigma^2}\left(r^2 + r\sigma^2 + \tfrac{1}{4}\sigma^4\right) = \frac{1}{\sigma^2}\left(r + \tfrac{1}{2}\sigma^2\right)^2.$$

Therefore,

$$-\mu + \sqrt{\mu^2 + 2\lambda} = \frac{1}{\sigma}\left(r - \tfrac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\left(r + \tfrac{1}{2}\sigma^2\right) = \frac{2r}{\sigma}.$$

Equation(8.3.3) implies

$$\widetilde{\mathbb{E}}\left[e^{-r\tau_L}\right] = \exp\left\{-\frac{1}{\sigma}\log\frac{x}{L}\cdot\frac{2r}{\sigma}\right\} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

The second line in(8.3.11) follows.