Stochastic Calculus For Finance - volume 2

- Section 7.3 Knock-out Barrier Options

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7.3.1 Up-and-Out Call

Our underlying risky asset is geometric Brownian motion $dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$ where $\widetilde{W}(t)$, 0 ≤ t ≤ T is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a European call, expiring at time T , with strike price K and up-and-out barrier B. We assume K < B; otherwise, the option must knock out in order to be in the money and hence could only pay off zero.

The solution to the stochastic differential equation for the asset price is

$S(t) = S(0)e^{\sigma\widetilde{W}(t)+\left(r-\frac{1}{2}\sigma^2\right)t} = S(0)e^{\sigma\widehat{W}(t)}$, where $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$ and $\alpha = \frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)$.

Define $\widehat{M}(T) = \max_{0\le t\le T} \widehat{W}(t)$,so $\max_{0\le t\le T} S(t) = S(0)e^{\sigma\widehat{M}(T)}$

The option knocks out if and only if $S(0)e^{\sigma\widehat{M}(T)} > B$

if $S(0)e^{\sigma\widehat{M}(T)} \le B$, $(S(T)-K)^+ = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)^+$ the payoff of the option is

$$V(T) = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)^+ \mathbb{I}_{\{S(0)e^{\sigma\widehat{M}(T)}\le B\}} = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)\mathbb{I}_{\{S(0)e^{\sigma\widehat{W}(T)}\ge K,\, S(0)e^{\sigma\widehat{M}(T)}\le B\}}$$

$$= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)\mathbb{I}_{\{\widehat{W}(T)\ge k,\, \widehat{M}(T)\le b\}}\,.\; k = \frac{1}{\sigma}\log\frac{K}{S(0)},\, b = \frac{1}{\sigma}\log\frac{B}{S(0)}$$

7.3.2 Black-Scholes-Merton Equation

Theorem 7.3.1.

Let $v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t) = x$. Then $v(t, x)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

in the rectangle $\{(t, x); 0 \le t < T, 0 \le x \le B\}$ and satisfies the boundary conditions

$$v(t, 0) = 0 \ , \ 0 \le t \le T$$

$$v(t, B) = 0 \ , \ 0 < t < T$$

$$v(T, x) = (x - K)^+ \ , 0 < x < B$$

The only exception to this is if the level B is first reached at the expiration time T ,for then there is no time left for the knock-out. In this case, the option price is given by the terminal condition (7.3.7). In particular, the function v(t, x) is not continuous at the corner of its domain where t= T and x = B. It is continuous everywhere elsein the rectangle $\{(t, x); 0 \le t < T, 0 \le x \le B\}$

Define the option payoff $V(T)$ by (7.3.2). The price of the option at time t between initiation and expiration is given by the risk-neutral pricing formula

$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T) \;\middle|\; \mathscr{F}(t)\right], 0 \le t \le T$. shows that $e^{-rt}V(t) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T) \;\middle|\; \mathscr{F}(t)\right], 0 \le t \le T$.

is a martingale

$v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t. $V(t)$ is the value of the option without any assumption

if the underlying asset price rises above the barrier B and then returns below the barrier by time t, then $V(t)$ will be zero because the option has knocked out, but $v(t, S(t))$ will be strictly positive

Defining ρ to be the first time t at which the asset price reaches the barrier B. In other words, ρ is chosen in a path-dependent way so that S(t) < B for $0 < t < \rho$ and $S(\rho) = B$.

proof:

Compute the differential

$$d\left(e^{-rt}v(t,S(t))\right)$$

$$= \frac{d}{dt}\left[e^{-rt}v(t,S(t))\right] + \frac{d}{dS}\left[e^{-rt}v(t,S(t))\right] + \frac{1}{2}\frac{d^2}{dS^2}\left[e^{-rt}v(t,S(t))\right]$$

$$= -re^{-rt}v(t,S(t))\,dt + e^{-rt}v_t(t,S(t))\,dt + e^{-rt}v_x(t,S(t))\,dS(t) + \frac{1}{2}e^{-rt}v_{xx}(t,S(t))\,(dS(t))^2$$

$$= e^{-rt}\left[-rv(t,S(t))\,dt + v_t(t,S(t))\,dt + v_x(t,S(t))\left(rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)\right) + \frac{1}{2}v_{xx}(t,S(t))\cdot\sigma^2 S^2(t)\,dt\right]$$

$$= e^{-rt}\left[-rv(t,S(t)) + v_t(t,S(t)) + rS(t)v_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)v_{xx}(t,S(t))\right]dt$$

$$+\, e^{-rt}\left[\sigma S(t)v_x(t,S(t))\right]d\tilde{W}(t)$$

($=0$: BS-equation)

The dt term must be zero for $0 \le t \le \rho$, (i.e., before the option knocks out). But since $(t, S(t))$ can reach any point in $\{(t,x); 0 \le t < T, 0 \le x \le B\}$ before the option knocks out, the Black-Scholes-Merton equation (7.3.4) must hold for every $t \in [0,T)$ and $x \in [0,B]$.

Lemma 7.3.2

We have

$$V(t) = v(t, S(t)), 0 \leq t \leq \rho.$$

In particular, $e^{-rt}v(t, S(t))$ is $\widetilde{\mathbb{P}}$-martingale up to time ρ, or, put another way, the stopped process

$$e^{-r(t\wedge\rho)}v(t \wedge \rho, S(t \wedge \rho)) \text{ , } 0 \leq t \leq T \qquad (7.3.12)$$

is a martingale under $\widetilde{\mathbb{P}}$

proof:

Because $v(t, S(t))$ is the value of the up-and-out call under the assumption that it has not knocked out before time t, and for $t < \rho$ this assumption is correct, we have (7.3.11) for $t < \rho$. From (7.3.11), we conclude that the process in (7.3.12) is the $\widetilde{\mathbb{P}}$-martingale (7.3.10).

Remark 7.3.3.

From Theorem 7.3.1 and its proof, we see how to construct a hedge, at least theoretically. Setting the dt term in (7.3.13) equal to zero, we obtain

$$d\left(e^{-rt}v(t,S(t))\right) = e^{-rt}\sigma S(t)v_x(t,S(t))\,d\widetilde{W}(t)$$

The discounted value of a portfolio that at each time t holds $\Delta(t)$ shares of the underlying asset is given by (see (5.2.27))

$$d\left(e^{-rt}X(t)\right) = e^{-rt}\sigma S(t)\Delta(t)\,d\widetilde{W}(t)$$

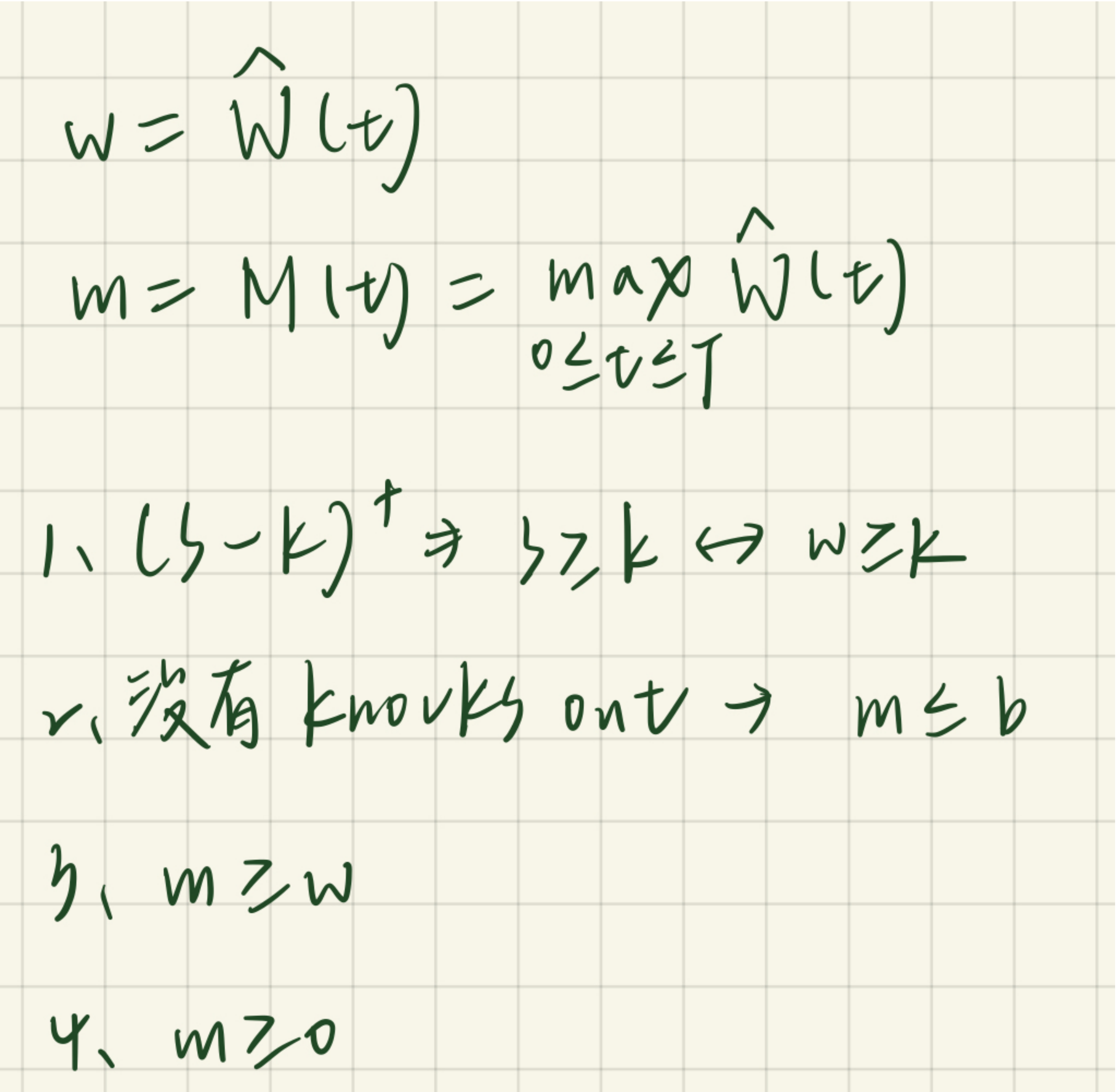
if an agent begins with a short position in the up-and-out call and with initial capital $X(0) = v(0,S(0))$, then the usual delta-hedging formula

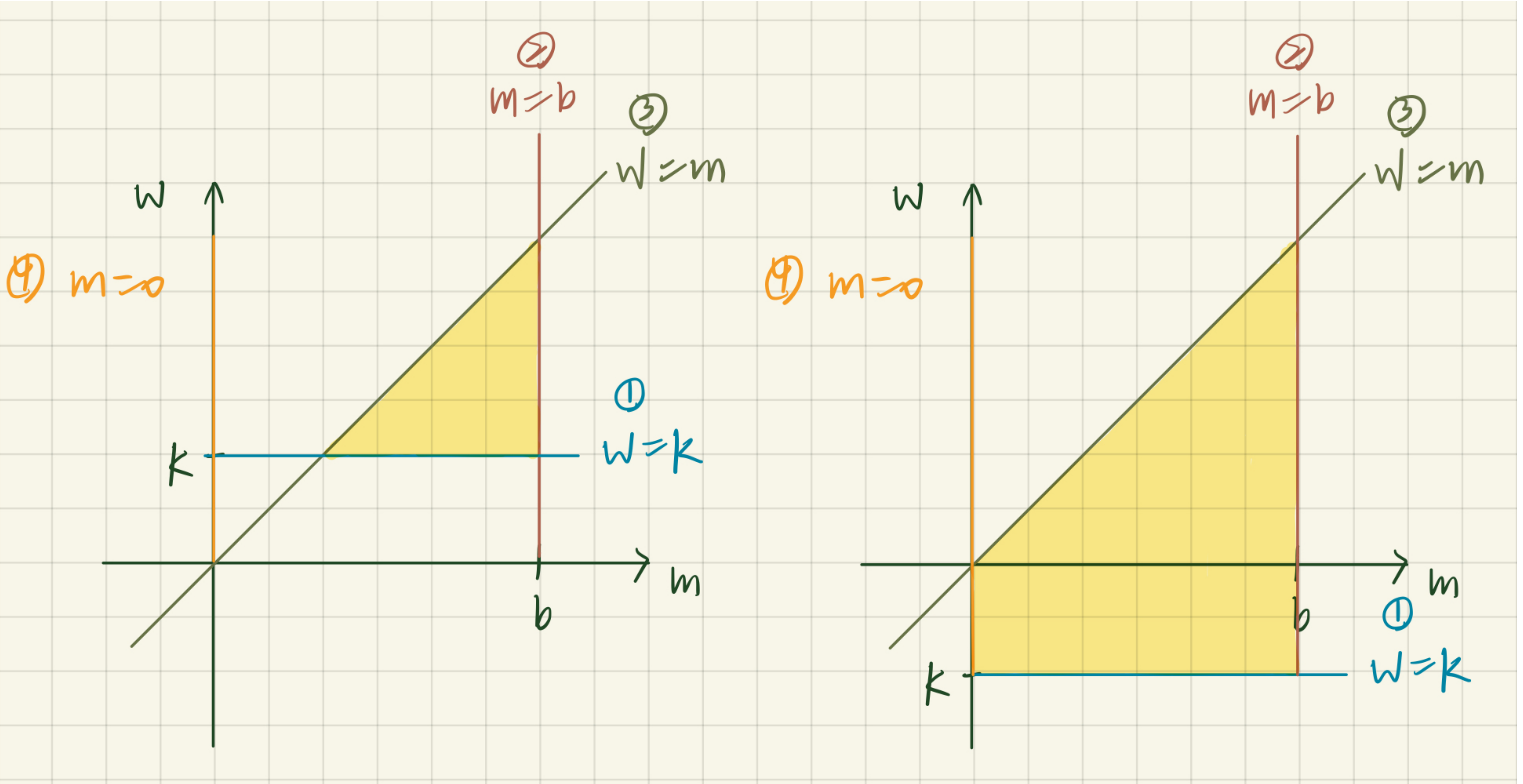
$$\Delta(t) = v_x(t,S(t))$$

will cause her portfolio value $X(t)$ to track the option value $v(t,S(t))$ up to the time ρ of knock-out or up to expiration T , whichever comes first.

7.3.3. Computation of the Price of the Up-and-Out Call

The risk-neutral price at time zero of the up-and-out call with payoff V (T) given by (7.3.2) is $V(0) = E[e^{-rT}V(T)]$. We use the density formula (7.2.3) to compute this.





$$V(0) = \int_k^b \int_{w^+}^b e^{-rT}\left(S(0)e^{\sigma w} - K\right) \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \, dm \, dw$$

by m chain rule

$$= -\int_k^b e^{-rT}\left(S(0)e^{\sigma w} - K\right) \frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \Big|_{m=w^+}^{m=b} dw$$

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$$= \frac{1}{\sqrt{2\pi T}} \int_k^b \left(S(0)e^{\sigma w} - K\right) e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} \, dw$$

$$- \frac{1}{\sqrt{2\pi T}} \int_k^b \left(S(0)e^{\sigma w} - K\right) e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2} \, dw$$

$$= S(0) I_1 - K I_2 - S(0) I_3 + K I_4$$

$$I_1 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} \, dw$$

$$I_2 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2} \, dw$$

$$I_3 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2} \, dw$$

$$I_4 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2} \, dw$$

可寫成 $\frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta + \gamma w - \frac{1}{2T}w^2} \, dw = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-\frac{1}{2T}(w-\gamma T)^2 + \frac{1}{2}\gamma^2 T + \beta} \, dw$ 配方法

$$= e^{\frac{1}{2}\gamma^2 T + \beta} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T}}(k-\gamma T)}^{\frac{1}{\sqrt{T}}(b-\gamma T)} e^{-\frac{1}{2}y^2} \, dy$$

$y = \frac{w-\gamma T}{\sqrt{T}}$ 變數變換

$dy = \frac{1}{\sqrt{T}} dw$

$$\frac{1}{\sqrt{2\pi T}}\int_k^b e^{\beta+\gamma w-\frac{w^2}{2T}}\,dw$$
$$= e^{\frac{1}{2}\gamma^2T+\beta}\left[N\left(\frac{b-\gamma T}{\sqrt{T}}\right)-N\left(\frac{k-\gamma T}{\sqrt{T}}\right)\right] = e^{\frac{1}{2}\gamma^2T+\beta}\left[N\left(\frac{-k+\gamma T}{\sqrt{T}}\right)-N\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right]$$
$$= e^{\frac{1}{2}\gamma^2T+\beta}\left[N\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{S(0)}{K}+\gamma\sigma T\right)\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{S(0)}{B}+\gamma\sigma T\right)\right)\right].$$

Let $\delta_{\pm}(\tau,s)=\frac{1}{\sigma\sqrt{\tau}}\left[\log s+\left(r\pm\frac{1}{2}\sigma^2\right)\tau\right]$. with $\beta=-rT-\frac{1}{2}\alpha^2T$ and $\gamma=\alpha+\sigma$, so $\frac{1}{2}\gamma^2T+\beta=0$ and $\gamma\sigma=r+\frac{1}{2}\sigma^2$.

$I_1=N\left(\delta_+\left(T,\frac{S(0)}{K}\right)\right)-N\left(\delta_+\left(T,\frac{S(0)}{B}\right)\right)$. $I_2=e^{-rT}\left[N\left(\delta_-\left(T,\frac{S(0)}{K}\right)\right)-N\left(\delta_-\left(T,\frac{S(0)}{B}\right)\right)\right]$.

$$I_3=\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}-1}\left[N\left(\delta_+\left(T,\frac{B^2}{KS(0)}\right)\right)-N\left(\delta_+\left(T,\frac{B}{S(0)}\right)\right)\right].$$

$$I_4=e^{-rT}\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T,\frac{B^2}{KS(0)}\right)\right)-N\left(\delta_-\left(T,\frac{B}{S(0)}\right)\right)\right].$$

$$I_1 = \frac{1}{\sqrt{2\pi T}} \int_K^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\sigma^2 T - \frac{1}{2T} w^2} \, dw$$

$$= \frac{1}{\sqrt{2\pi T}} \int_K^b e^{(-rT - \frac{1}{2}\sigma^2 T) + (\sigma + \alpha) w - \frac{1}{2T} w^2} \, dw$$

$$rT + \frac{1}{2}\sigma^2 T$$

$$\frac{1}{\sqrt{2\pi T}} \int_K^b e^{\beta + \gamma w - \frac{1}{2T} w^2} \, dw = e^{\frac{1}{2}\gamma^2 T + \beta} \left[N\left(\frac{1}{\sigma\sqrt{T}} \left[\log \frac{S(0)}{K} + \gamma\sigma T \right] \right) - N\left(\frac{1}{\sigma\sqrt{T}} \left[\log \frac{S(0)}{B} + \gamma\sigma T \right] \right) \right]$$

$$= 0$$

$$\delta_{\pm}(\tau, s) = \frac{1}{\sigma\sqrt{\tau}} \left[\log s + \left(r \pm \frac{1}{2}\sigma^2 \right) \tau \right]$$

$$\Rightarrow I_1 = N\left(\delta_+\left(T, \frac{S(0)}{K}\right) \right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right) \right)$$

$$
\begin{aligned}
V(0) = \; & S(0)\left[N\left(\delta_+\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right)\right)\right. \\
& -e^{-rT}K\left[N\left(\delta_-\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_-\left(T, \frac{S(0)}{B}\right)\right)\right. \\
& -B\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_+\left(T, \frac{B}{S(0)}\right)\right)\right. \\
& +e^{-rT}K\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_-\left(T, \frac{B}{S(0)}\right)\right)\right.
\end{aligned}
$$

Let $t \in [0,T)$,and assume the underlying asset price at time t is $S(t) = x$. As above, we assume $0 < x \leq B$. If the call has not knocked out prior to time t, its price at time t is obtained by replacing T by the time to expiration $\tau = T - t$
and replacing S(0) by x

$$v(t,x) = \ x\left[N\left(\delta_+\left(\tau, \frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau, \frac{x}{B}\right)\right)\right.$$

$$-e^{-r\tau}K\left[N\left(\delta_-\left(\tau, \frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau, \frac{x}{B}\right)\right)\right.$$

$$-B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau, \frac{B}{x}\right)\right)\right]$$

$$+e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau, \frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau, \frac{B}{x}\right)\right)\right], 0 \leq t < T \ , 0 < x \leq B$$