Stochastic Calculus For Finance - volume 2

- Section 5.4.3 Existence of the Risk-Neutral Measure
- Section 5.4.4 Uniqueness of the Risk-Neutral Measure

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Definition 5.4.3.

A probability measure $\widetilde{\mathbb{P}}$ is said to be risk-neutral if

(i) $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e., for every $A \in \mathscr{F}$, $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$),

(ii) under $\widetilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, \ldots, m$.

To make discounted stock prices be martingales, we would like to rewrite (5.4.15) as (5.4.16)

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t)[(\alpha_i(t) - R(t))\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)] \ldots (5.4.15)$$

$$= D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)[\Theta_j(t)\,dt + dW_j(t)] \ldots (5.4.16)$$

If we can find the market price of risk processes $\Theta_j(t)$ (multiple dimensions) that make (5.4.16) hold,

By Girsanov theorem We can construct an equivalent probability measure $\widetilde{\mathbb{P}}$ under $\widetilde{W}(t)$

$$\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u)\,\mathrm{d}u \rightarrow \mathrm{d}\widetilde{W}_j(t) = \mathrm{d}W_j(t) + \Theta_j(t)\mathrm{d}t$$

and we can get $d\big(D(t)S_i(t)\big) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\,d\widetilde{W}_j(t)$. Hence $D(t)S_i(t)$ is a martingale under $\widetilde{\mathbb{P}}$.

To find a risk-neutral measure is simply one of finding processes $\Theta_j(t)$ that make (5.4.15) and (5.4.16) agree. Since these equations have the same coefficient multiplying each $dW_j(t)$, they agree if and only if the coefficient multiplying dt is the same in both cases, which means that

$$\alpha_i(t) - R(t) \;=\; \sum_{j=1}^{d}\sigma_{ij}(t)\,\Theta_j(t), \quad i = 1,\ldots,m\,.$$

We call these the market price of risk equations. These are m equations in the d unknown processes $\Theta(t) = \Theta_1(t),\ldots,\Theta_d(t)$

In the market with stock prices $S_i(t)$ and interest rate process $R(t)$, an agent can begin with initial capital X(0) and choose adapted portfolio processes $\Delta_i(t)$, one for each stock $S_i(t)$. The differential of the agent's portfolio value will then be $dX(t) = R(t)X(t)\,dt + \sum_{i=1}^{m} \Delta_i(t)\big(dS_i(t) - R(t)S_i(t)\,dt\big) = R(t)X(t)\,dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)}\,d\big(D(t)S_i(t)\big)$. The differential of the discounted portfolio value is

$d\big(D(t)X(t)\big) = D(t)\big(dX(t) - R(t)X(t)\,dt\big) = \sum_{i=1}^{m} \Delta_i(t)\,d(D(t)S_i(t)$.

If $\widetilde{\mathbb{P}}$ is a risk-neutral measure, then under $\widetilde{\mathbb{P}}$ the processes $D(t)S(t)$ are martingales, and hence the process $D(t)X(t)$ must also be a martingale.

Lemma 5.4.5 Let $\widetilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\widetilde{\mathbb{P}}$, the discounted portfolio value $D(t)X(t)$ is a martingale.

投組價值（單一股票） $dX(t) = \Delta(t)\,dS(t) + R(t)\left[X(t) - \Delta(t)S(t)\right]dt$

m 支股票 $dX(t) = \sum_{i=1}^{m}\Delta_i(t)\,dS_i(t) + R(t)\left[X(t) - \sum_{i=1}^{m}\Delta_i(t)S_i(t)\right]dt$

$$= R(t)X(t)\,dt + \sum_{i=1}^{m}\Delta_i(t)\left[dS_i(t) - R(t)S_i(t)\,dt\right]$$

$$= R(t)X(t)\,dt + \sum_{i=1}^{m}\frac{\Delta_i(t)}{D(t)}\,d\left[D(t)S_i(t)\right]$$

☆貼現因子： $dD(t) = -R(t)D(t)\,dt$

$$d\left[D(t)S_i(t)\right] = dD(t)\,S_i(t) + D(t)\,dS_i(t)$$

$$= D(t)\,dS_i(t) - R(t)D(t)S_i(t)\,dt$$

同除 $D(t) \rightarrow dS_i(t) - R(t)S_i(t)\,dt = \frac{1}{D(t)}\,d\left[D(t)S_i(t)\right]$

$$d\left(D(t)X(t)\right) = dD(t)X(t) + D(t)\,dX(t)$$

$$= -R(t)D(t)\cancel{X(t)}\,dt + D(t)\left\{R(t)\cancel{X(t)}\,dt + \sum_{i=1}^{m}\frac{\Delta_i(t)}{D(t)}\,d\left[D(t)S_i(t)\right]\right\}$$

$$= \sum_{i=1}^{m}\Delta_i(t)\,d\left[D(t)S_i(t)\right]$$

Definition 5.4.6.

An arbitrage is a portfolio value process X(t) satisfying X(0) = 0 and also satisfying for some time T > 0

$$\mathbb{P}\Big\{X(T) \geq 0\Big\} = 1, \qquad \mathbb{P}\Big\{X(T) > 0\Big\} > 0.$$

In other words, there exists an arbitrage if and only if there is a way to start with $X(0)$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\left\{X(T) \geq \frac{X(0)}{D(T)}\right\} = 1, \qquad \mathbb{P}\left\{X(T) > \frac{X(0)}{D(T)}\right\} > 0.$$

Theorem 5.4.7 ***(First fundamental theorem of asset pricing).***

If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

proof:

If a market model has a risk-neutral probability measure $\widetilde{\mathbb{P}}$, by **Lemma 5.4.5** every discounted portfolio value process is a martingale under $\widetilde{\mathbb{P}}$. In particular, every portfolio value process satisfies $\widetilde{\mathbb{E}}[D(T)X(T)] = X(0)$. Let $X(t)$ be a portfolio value process with $X(0) = 0$. Then we have $\widetilde{\mathbb{E}}[D(T)X(T)] = 0$.

Suppose $X(T)$ satisfies $\mathbb{P}\{X(T) > 0\} > 0$ Since $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, we have also $\widetilde{\mathbb{P}}\{X(T) > 0\} > 0$.Then $\widetilde{\mathbb{E}}[D(T)X(T)] > 0$ contradicts with $\widetilde{\mathbb{E}}[D(T)X(T)] = 0$. Hence, $X(t)$ is not an arbitrage. In fact, there cannot exist an arbitrage since every portfolio value process $X(t)$ satisfying $X(0) = 0$ cannot be an arbitrage.

5.4.4 Uniqueness of the Risk-Neutral Measure

Definition 5.4.8.

A market model is complete if every derivative security can be hedged.

Suppose we have a market model with a filtration generated by a d-dimensional Brownian motion and with a risk-neutral measure $\widetilde{\mathbb{P}}$, used the resulting market prices of risk $\Theta_1(t), \ldots, \Theta_d(t)$ to define the Radon-Nikodym derivative process $Z(t)$, and have changed to the measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}(t)$ defined by (5.4.2) is a d-dimensional Brownian motion). Suppose further that we are given an $\mathscr{F}(T) - \mathit{measurable}$ random variable $V(T)$, which is the payoff of some derivative security. $V(t) = \widetilde{\mathbb{E}}[\, e^{-\int_t^T R(u)\,\mathrm{d}u}\, V(T) \mid \mathscr{F}(t)]\,, 0 \le t \le T$ satisfies

$$D(t)\,V(t) = \widetilde{\mathbb{E}}\left[D(T)\,V(T) \mid \mathscr{F}(t)\right] = V(0)\; + \;\sum_{j=1}^{d}\int_0^t \widetilde{\Gamma}_j(u)\,\mathrm{d}\widetilde{W}_j(u)$$

by (5.4.22) and (5.4.17),

$$d\big(D(t)X(t)\big) = \sum_{i=1}^{m} \Delta_i(t)\, d\big(D(t)S_i(t)\big) = \sum_{j=1}^{d}\sum_{i=1}^{m} \Delta_i(t)\, D(t)S_i(t)\, \sigma_{ij}(t)\, d\widetilde{W}_j(t),$$

or, equivalently, $D(t)X(t) = X(0) + \sum_{j=1}^{d} \int_0^t \sum_{i=1}^{m} \Delta_i(u)\, D(u)S_i(u)\, \sigma_{ij}(u)\, d\widetilde{W}_j(u),$

to hedge the short position, we should take X(0) = V(0) and choose the portfolio processes $\Delta_1(t), \ldots, \Delta_m(t)$ so that the hedging equations

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t)\, S_i(t)\, \sigma_{ij}(t), j = 1,...,d$$

are satisfied. These are d equations in m unknown processes $\Delta_1(t), \ldots, \Delta_m(t)$

Theorem 5.4.9 ***(Second fundamental theorem of asset pricing).***

Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

proof:

$\Rightarrow$) Assume that the model is complete. We wish to show that there can be only one risk-neutral measure. Suppose the model has two risk-neutral measures, $\widetilde{\mathbb{P}_1}$ and $\widetilde{\mathbb{P}_2}$ .Let A be a set in $\mathscr{F}$, consider the derivative security with payoff $V(T) = \mathbb{I}_A \frac{1}{D(T)}$ .Because the model is complete, a short position in this derivative security can be hedged .Since both $\widetilde{\mathbb{P}_1}$ and $\widetilde{\mathbb{P}_2}$ are risk-neutral, the discounted portfolio value process $D(t)X(t)$ is a martingale under both $\widetilde{\mathbb{P}_1}$ and $\widetilde{\mathbb{P}_2}$. It follows that

$$\widetilde{\mathbb{P}}_1(A) = \widetilde{\mathbb{E}}_1\left[D(T)V(T)\right] = \widetilde{\mathbb{E}}_1\left[D(T)X(T)\right] = X(0) = \widetilde{\mathbb{E}}_2\left[D(T)X(T)\right] = \widetilde{\mathbb{E}}_2\left[D(T)V(T)\right] = \widetilde{\mathbb{P}}_2(A).$$

proof:

$\Leftarrow$) Suppose there is only one risk-neutral measure.implies that the market price of risk equations (5.4.18) have only one solution $\Theta_1(t), \ldots, \Theta_d(t)$

For fixed t and w , these equations are of the form $\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\, \Theta_j(t), \qquad i = 1, \ldots, m.$

$$\begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}(t) & \sigma_{m2}(t) & \cdots & \sigma_{md}(t) \end{bmatrix} \begin{bmatrix} \Theta_1(t) \\ \Theta_2(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix} = \begin{bmatrix} \alpha_1(t) - R(t) \\ \alpha_2(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix}$$

In order to be assured that every derivative security can be hedged(Def:complete), we must be able to solve the hedging equations (5.4.29) for $\Delta_1(t), \ldots, \Delta_m(t)$

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t)\, S_i(t)\, \sigma_{ij}(t), \qquad j = 1, \ldots, d.$$

proof:

no matter what values of $\dfrac{\widetilde{\Gamma}(t)}{D(t)}$

$D(t)$ appear on the left-hand side. For fixed t and w, the hedging equations are of the

$$\begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}(t) & \sigma_{m2}(t) & \cdots & \sigma_{md}(t) \end{bmatrix}^T \begin{bmatrix} \Delta_1(t)\,S_1(t) \\ \Delta_2(t)\,S_2(t) \\ \vdots \\ \Delta_m(t)\,S_m(t) \end{bmatrix} = \begin{bmatrix} \widetilde{\Gamma}_1(t)/D(t) \\ \widetilde{\Gamma}_2(t)/D(t) \\ \vdots \\ \widetilde{\Gamma}_d(t)/D(t) \end{bmatrix}$$

In order to be assured that the market is complete, there must be a solution to the system of equations.The uniqueness of the solution to (5.4.30) implies the existence of a solution y to (5.4.32). Consequently, uniqueness of the risk-neutral measure implies that the market model is complete.