Stochastic Calculus For Finance - volume 2

- Section 4.7.1 Gaussian Processes
- Section 4.7.2 Brownian Bridge as a Gaussian Process
- Section 4.7.3 Brownian Bridge as a Scaled Stochastic Integral

25/04/22, 羅芷羚

Theorem 4.6.5 (Levy, two dimensions)

Let $M_1(t)$ and $M_2(t)$, $t \geq 0$, be martingales relative to a filtration $\mathscr{F}(t)$, $t \geq 0$.
Assume that for $i = 1,2$, we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1, M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.

Recall Theorem 4.6.4 (Levy, one dimension)

Let $M(t)$,$t \geq 0$, be a martingale relative to a filtration $\mathscr{F}(t)$, $t \geq 0$. Assume that ① $M(0) = 0$, $M(t)$ has ② continuous paths, and ③ $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.

$\Rightarrow$ $M_1(t)$ and $M_2(t)$ are Brownian motions.

proof:

To show independence, we examine the joint moment-generating function.

Let $f(t, x, y)$ be a function whose derivatives are defined and continuous. The two-dimensional Itô-Doeblin formula implies that

$$df(t, M_1, M_2) = f_t\,dt + f_x\,dM_1 + f_y\,dM_2 + \frac{1}{2}f_{xx}\underbrace{\boxed{dM_1 dM_1}}_{dt} + \frac{1}{2}f_{yy}\underbrace{\boxed{dM_2 dM_2}}_{dt} + \underbrace{\boxed{f_{xy}\,dM_1\,dM_2}}_{0}$$

$$= f_t\,dt + f_x\,dM_1 + f_y\,dM_2 + \frac{1}{2}f_{xx}\,dt + \frac{1}{2}f_{yy}\,dt$$

proof:

$$\mathbb{E}\int_0^t [f_t(s, M_1(s), M_2(s)) + \frac{1}{2} f_{xx}(s, M_1(s), M_2(s)) + \frac{1}{2} f_{yy}(s, M_1(s), M_2(s))]\, ds$$

$$= \mathbb{E}\int_0^t [-\frac{1}{2}(u_1^2 + u_2^2) f + \frac{1}{2} u_1^2 f + \frac{1}{2} u_2^2 f] = 0$$

$$f = \exp[u_1 x + u_2 y - \frac{1}{2}(u_1^2 + u_2^2) t]$$

$$f_t = [-\frac{1}{2}(u_1^2 + u_2^2)] f \quad f_{xx} = u_1^2 f \quad f_{yy} = u_2^2 f.$$

We integrate both sides to obtain

$$f\left(t, M_1(t), M_2(t)\right)$$
$$= f\left(0, M_1(0), M_2(0)\right) + \int_0^t \left[f_t\left(s, M_1(s), M_2(s)\right) + \frac{1}{2} f_{xx}\left(s, M_1(s), M_2(s)\right) + \frac{1}{2} f_{yy}\left(s, M_1(s), M_2(s)\right) \right] ds$$
$$+ \int_0^t f_x\left(s, M_1(s), M_2(s)\right) dM_1(s) + \int_0^t f_y\left(s, M_1(s), M_2(s)\right) dM_2(s)$$

Taking expectations on both sides

$$\mathbb{E} f\left(t, M_1(t), M_2(t)\right)$$
$$= f\left(0, M_1(0), M_2(0)\right) + \mathbb{E}\int_0^t [\, f_t\left(s, M_1(s), M_2(s)\right) + \frac{1}{2} f_{xx}\left(s, M_1(s), M_2(s)\right)$$
$$+ \frac{1}{2} f_{yy}\left(s, M_1(s), M_2(s)\right)] ds$$

proof:

Now, fix numbers u_1 and u_2 and define

$$f(t,x,y) = \exp\left\{u_1x + u_2y - \frac{1}{2}(u_1^2 + u_2^2)t\right\}$$

Then $f_t(t,x,y) = -\frac{1}{2}(u_1^2 + u_2^2)f(t,x,y)$, $f_x(t,x,y) = u_1 f(t,x,y)$, $f_y(t,x,y) = u_2 f(t,x,y)$

$f_{xx}(t,x,y) = u_1^2 f(t,x,y)$, $f_{yy}(t,x,y) = u_2^2 f(t,x,y)$

the second term on the right-hand side is zero

$$\mathbb{E}\exp\left\{u_1M_1(t) + u_2M_2(t) - \frac{1}{2}(u_1^2 + u_2^2)t\right\} = 1$$

$$\Rightarrow \mathbb{E}\exp\left\{u_1M_1(t) + u_2M_2(t)\right\} = \exp\left\{\frac{1}{2}(u_1^2 + u_2^2)t\right\} = exp\left\{\frac{1}{2}u_1^2t\right\} exp\left\{\frac{1}{2}u_2^2t\right\}$$

$= \mathbb{E}[e^{u_1M_1(t)}] \times \mathbb{E}[e^{u_2M_2(t)}] \Rightarrow M_1(t), M_2(t)$ must be independent.

4.7 Brownian Bridge

This is a stochastic process that is like a Brownian motion except that with probability one. It reaches a specified point at a specified positive time. We first discuss Gaussian processes in general, the class to which the Brownian bridge belongs, and we then define the Brownian bridge and present its properties. The primary use for the Brownian bridge in finance is as an aid to Monte Carlo simulation.

4.7.1 Gaussian Processes

Definition 4.7.1.

A Gaussian process $X(t)$, $t > 0$, is a stochastic process that has

the property that, for arbitrary times $0 < t_1 < t_2 < \ldots < t_n$, the random variables

$X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.

The joint normal distribution of a set of vectors is determined by their means and covariances. Therefore,

for a Gaussian process, the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is determined by the means and

covariances of these random variables.

We denote the mean of X(t) by m(t), $m(t) = \mathbb{E}X(t)$

And denote the covariance of X(s) and X(t) by c(s, t) = E$c(s, t) = \mathbb{E}[(X(s) - m(s))(X(t) - m(t))]$.

4.7.2 Brownian Bridge as a Gaussian Process

Definition 4.7.4.

Let $W(t)$ be a Brownian motion. Fix $T > 0$. We define the Brownian bridge from 0 to 0 on [0, T] to be the process

$$X(t) = W(t) - \frac{t}{T}W(T) \text{ , } 0 < t < T \text{ .}$$

Note that $\frac{t}{T}W(T)$ as a function of t is the line from (0, 0) to (T,W(T))

We have subtracted this line away from the Brownian motion W(t), the resulting process X(t) satisfies X(0) = X(T) = 0

Recall 3.3.2

Because the increments of brownian motion are independent and normally distributed, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed.

For $0 < t_1 < t_2 < \ldots < t_n < T$, the random variables

$X(t_1) = W(t_1) - \frac{t_1}{T}W(T), \ldots, X(t_n) = W(t_n) - \frac{t_n}{T}W(T)$ are jointly normal because $W(t_1), \ldots, W(t_n), W(T)$ are jointly normal. Hence, the Brownian bridge from 0 to 0 is a Gaussian process.

so, the function is easily seen to be

$$m(t) = \mathbb{E}X(t) = \mathbb{E}\left[W(t) - \frac{t}{T}W(T)\right] = 0 \quad ,$$

$$c(s,t) = \mathbb{E}\left[\left(W(s) - \frac{s}{T}W(T)\right)\left(W(t) - \frac{t}{T}W(T)\right)\right]$$

$$= \mathbb{E}[W(s)W(t)] - \frac{t}{T}\mathbb{E}[W(s)W(T)] - \frac{s}{T}\mathbb{E}[W(t)W(T)] + \frac{st}{T^2}\mathbb{E}[W^2(T)]$$

$$= s \wedge t - \frac{2st}{T} + \frac{st}{T} = s \wedge t - \frac{st}{T}$$

Recall 3.3.2

- The covariance matrix for Brownian motion (i.e., for the m-dimensional random vector $\left(W(t_1), W(t_2), \cdots, W(t_m)\right)$) is

$$\begin{bmatrix} \mathrm{E}\left[W^2(t_1)\right] & \mathrm{E}\left[W(t_1)W(t_2)\right] & \cdots & \mathrm{E}\left[W(t_1)W(t_m)\right] \\ \mathrm{E}\left[W(t_2)W(t_1)\right] & \mathrm{E}\left[W^2(t_2)\right] & \cdots & \mathrm{E}\left[W(t_2)W(t_m)\right] \\ \vdots & \vdots & & \vdots \\ \mathrm{E}\left[W(t_m)W(t_1)\right] & \mathrm{E}\left[W(t_m)W(t_2)\right] & \cdots & \mathrm{E}\left[W^2(t_m)\right] \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

Definition 4.7.5.

Let W(t) be a Brownian motion. Fix T > 0, $a \in R$, and $b \in R$. We define the Brownian bridge from a to b on [0, T] to be the process

$$X^{a\to b}(t) = a + \frac{(b-a)t}{T} + X(t), \qquad 0 \le t \le T,$$

The mean function is affected

$$m^{a\to b}(t) = \mathbb{E}X^{a\to b}(t) = a + \frac{(b-a)t}{T}.$$

the covariance function is not affected

$$c^{a\to b}(s,t) = \mathbb{E}\left[\left(X^{a\to b}(s) - m^{a\to b}(s)\right)\left(X^{a\to b}(t) - m^{a\to b}(t)\right)\right] = s \wedge t - \frac{st}{T}.$$

$$= \mathbb{E}\Big[\Big(\underbrace{\Big(a + \frac{(b-a)s}{T} + X(s)\Big)}_{X^{a\to b}(s)} - \underbrace{\Big(a + \frac{(b-a)s}{T}\Big)}_{m^{a\to b}(s)}\Big)\Big(\underbrace{\Big(a + \frac{(b-a)t}{T} + X(t)\Big)}_{X^{a\to b}(t)} - \underbrace{\Big(a + \frac{(b-a)t}{T}\Big)}_{m^{a\to b}(t)}\Big)\Big]$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

We cannot write the Brownian bridge as a stochastic integral of a deterministic integrand because the variance of the Brownian bridge isn't monotonically increasing

$$\mathbb{E}X^2(t) = c(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$$

Obtain a process with the same distribution as the Brownian bridge from 0 to 0 as a scaled stochastic integral. In particular, consider

$$Y(t) = (T-t)\int_0^t \frac{1}{T-u}\, dW(u), \quad 0 \le t < T.$$

The integral

$$I(t) = \int_0^t \boxed{\frac{1}{T-u}}\, dW(u) \text{ is a Gaussian process}$$

Recall Theorem 4.4.9 (Itô integral of a deterministic integrand).

Let $W(s)$, $s \geq 0$, be a Brownian a motion, and let $\Delta(s)$ be a nonrandom function of time.

Define $I(t) = \int_0^t \Delta(s)dW(s)$. For each $t \geq 0$, the random variable I(t) is normally distributed

with expected value zero and variance $Var\, I(t) = \int_0^t \Delta^2(s)ds$

For $0 < t_1 < t_2 < \ldots < t_n < T$, the random variables

$Y(t_1) = (T - t_1)I(t_1), Y(t_2) = (T - t_2)I(t_2), \ldots, Y(t_n) = (T - t_n)I(t_n)$

are jointly normal because $I(t_1), I(t_2), \ldots, I(t_n)$ are jointly normal. In particular, Y is a Gaussian process.

The mean and covariance functions of I

$$m^I(t) = 0 \ , c^I(s,t) = \int_0^{s\wedge t} \frac{1}{(T-u)^2}\, du = \frac{1}{T - s\wedge t} - \frac{1}{T} \ \textit{for all } s,t \in [0,T)$$

so that $m^Y(t) = 0$, we assume for the moment that $0 \le s \le t < T$

$$c^Y(s,t) = \mathbb{E}[Y(s)Y(t)] = \mathbb{E}[(T-s)(T-t)I(s)I(t)] = (T-s)(T-t)\cdot c^I(s,t)$$

$$= (T-s)(T-t)\cdot\frac{s}{T(T-s)} = \frac{(T-t)s}{T} = s - \frac{st}{T}$$

If we had taken $0 \le s, t < T$

$c^Y(s,t) = s\wedge t - \dfrac{st}{T}$ for all $s,t \in [0,T)$

Theorem 4.7.6.

Define the process

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u}\,dW(u), & \text{for } 0 \le t < T, \\ 0, & \text{for } t = T. \end{cases}$$

Then Y (t) is a continuous Gaussian process on [0, T] and has mean and covariance functions.

$$m^Y(t) = 0, \quad t \in [0,T]$$

$$c^Y(s,t) = s \wedge t - \frac{st}{T} \quad \text{for all } s,t \in [0,T]$$

In particular, the process Y (t) has the same distribution as the Brownian bridge from 0 to 0 on [0, T]

Corollary 4.6.3 (Itô product rule)
Let X(t) and Y (t) be Itô processes. Then
$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$.

compute the stochastic differential of $Y(t)$,

$$dY(t) = \int_0^t \frac{1}{T-u}\, dW(u) \cdot d(T-t) + (T-t) \cdot d\left(\int_0^t \frac{1}{T-u}\, dW(u) \right)$$

$$= -\int_0^t \frac{1}{T-u}\, dW(u) \cdot dt + dW(t) = -\frac{Y(t)}{T-t}dt + dW(t)$$

If $Y(t) > 0$ as $t \to T$, the drift term $-\frac{Y(t)}{T-t}dt$ becomes large in absolute value and is negative. This drives $Y(t)$ toward zero.

On the other hand, if $Y(t) < 0$, the drift term becomes large and positive, and this again drives $Y(t)$ toward zero.

This strongly suggests, and it is indeed true, that as $t \to T$ the process $Y(t)$ converges to zero almost surely.