Stochastic Calculus For Finance - volume 2

Section 4.2 Itô's Integral for Simple Integrands

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Section 4.2 Itô's Integral for Simple Integrands

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$$\int_0^T \Delta(t)\, dW(t)$$

1.Brownian motion

2.Together with a filtration $\mathcal{F}(t)$

1.Adapted to $\mathcal{F}(t) \rightarrow \Delta(t)$ is $\mathcal{F}(t)$-measurable

2.$\Delta(t)$ is independent of future Brownian increments

3.At time t the randomness of $\Delta(t)$ has been resolved

$$\int_0^T \Delta(t)\, dg(t) = \int_0^T \Delta(t) g'(t)\, dt$$

Problem :

Brownian motion paths cannot be differentiated with respect to time.

4.2.1 Construction of the Integral

Define the Itô integral for simple integrands Δ(t)

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$;

i.e., $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$.

Assume that Δ(t) is constant in t on each subinterval $[t_j, t_{j+1})$.

Such a process Δ(t) is a simple process.

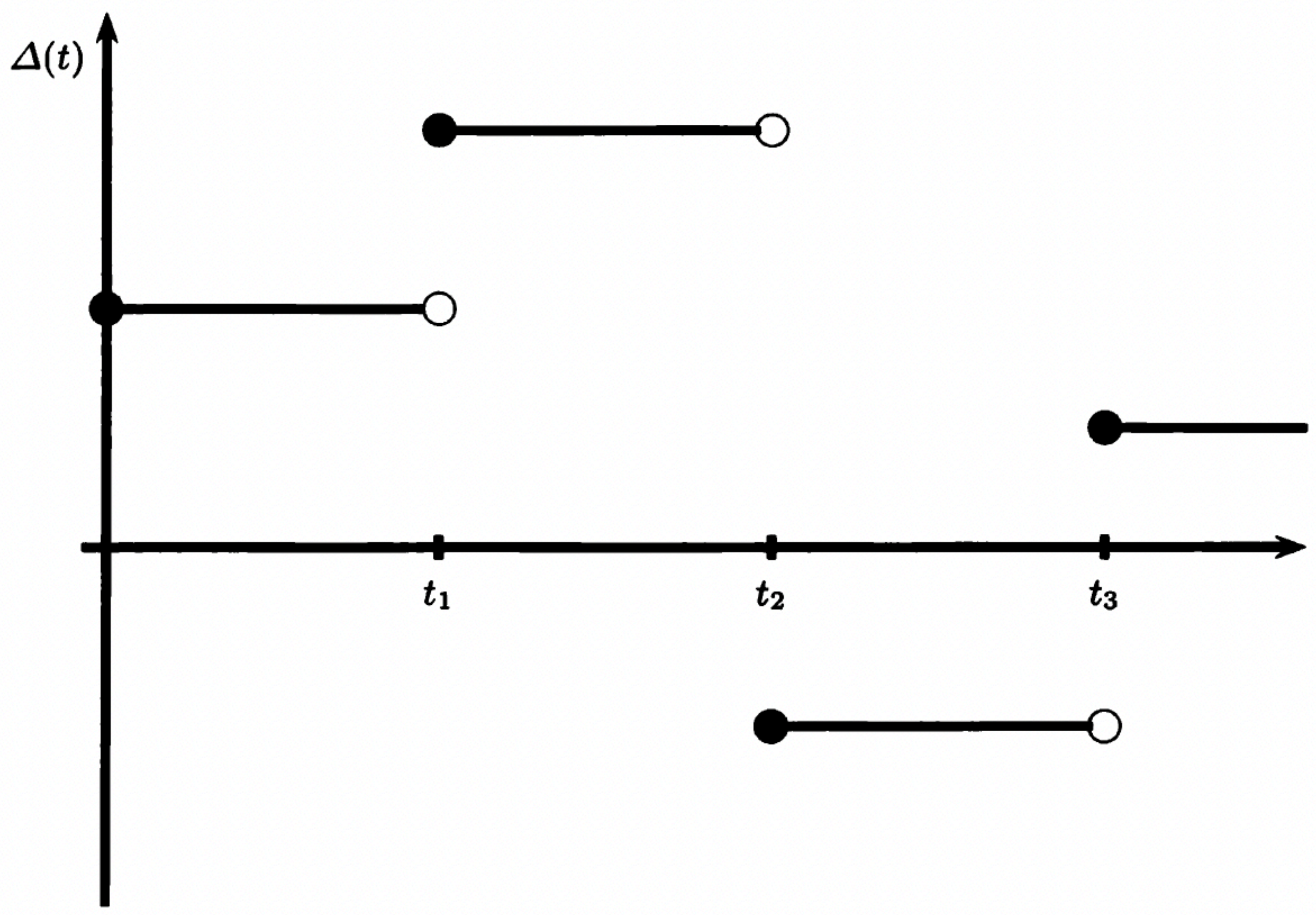

Fig. 4.2.1. A path of a simple process.

Explain from the trading behavior of financial markets

W(t) : the price per share of an asset at time t.

Δ(t): as the position taken in the asset at each trading date and held to the next trading date.

$t_0, t_1, \ldots, t_{n-1}$: as the trading dates in the asset.

I(t): the gain from trading at each time t.

$$I(t) = \Delta(t_0)[W(t) - W(t_0)] = \Delta(0)W(t),\ \ 0 \le t \le t_1,$$

$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)],\ \ t_1 \le t \le t_2,$$

$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t_2 - W(t_1] + \Delta(t_2)[W(t) - W(t_2)],\ \ t_2 \le t \le t_3,$$

$$In\ general, if\ t_k \le t \le t_{k+1}$$

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \rightarrow\ I(t) = \int_0^T \Delta(u)\, dW(u)$$

discrete continuous

Theorem 4.2.1. The Itô integral defined by (4.2.2) is a martingale.

$$\mathbb{E}[I(t) \mid \mathscr{F}(s)] = I(s) \quad \forall s \le t$$

proof :

Let $0 \le s \le t \le T$ be given. We shall assume that s and t are in different subintervals of the partition Π (i.e., there are partition points t_ℓ and t_k such that $t_\ell < t_k$, $s \in [t_\ell, t_{\ell+1})$ and $t \in [t_k, t_{k+1})$. If s and t are in the same subinterval, the following proof simplifies. Equation (4.2.2) may be rewritten as

①　②

$$I(t) = \underbrace{\sum_{j=0}^{\ell-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]}_{①} + \underbrace{\Delta(t_\ell)[W(t_{\ell+1}) - W(t_\ell)]}_{②}$$

0~s 的部分

$$+ \underbrace{\sum_{j=\ell+1}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]}_{③} + \underbrace{\Delta(t_k)[W(t) - W(t_k)]}_{④}$$

s~t 的部分

③　④

proof :

The latest time appearing in this sum is t_ℓ and $t_\ell \leq s$,so every random variable in the first sum is $\mathscr{F}(s) - measurable$. Therefore,

$$① \quad \mathbb{E}\left[\sum_{j=0}^{\ell-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] \,\middle|\, \mathscr{F}(s)\right] = \sum_{j=0}^{\ell-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$$

For the second term, we “take out what is known”and use the martingale property of W to write

By Theorem 3.3.4. Brownian motion is a martingale

$$② \quad \mathbb{E}\left[\Delta(t_\ell)\big(W(t_{\ell+1}) - W(t_\ell)\big) \,\middle|\, \mathscr{F}(s)\right] = \Delta(t_\ell)\Big(\mathbb{E}\left[W(t_{\ell+1}) \,\middle|\, \mathscr{F}(s)\right] - W(t_\ell)\Big)$$

$$= \Delta(t_\ell)\big(W(s) - W(t_\ell)\big)$$

proof :

The summands in the third term are of the form $\Delta(t_j)[W(t_{j+1}) - W(t_j)]$,where $t_j \geq t_{\ell+1} \geq s$. This permits us to use the following iterated conditioning trick, which is based on properties (iii) iterated conditioning and (ii) taking out what is known

③ $$\mathbb{E}\left\{\Delta(t_j)\Big(W(t_{j+1}) - W(t_j)\Big)\Big|\mathscr{F}(s)\right\} = \mathbb{E}\left\{\mathbb{E}\left[\Delta(t_j)\Big(W(t_{j+1}) - W(t_j)\Big)\Big|\mathscr{F}(t_j)\right]\Big|\mathscr{F}(s)\right\}$$

$$= \mathbb{E}\left\{\Delta(t_j)\Big(\mathbb{E}\left[W(t_{j+1})\Big|\mathscr{F}(t_j)\right] - W(t_j)\Big)\Big|\mathscr{F}(s)\right\}$$

$$= \mathbb{E}\left\{\Delta(t_j)\Big(W(t_j) - W(t_j)\Big)\Big|\mathscr{F}(s)\right\} = 0$$

※ iterated conditioning:
$\forall I_1 \subseteq I_2 , \mathbb{E}[\mathbb{E}(Y \mid I_2) \mid I_1] = \mathbb{E}(Y \mid I_1)$

$$\Rightarrow \mathbb{E}\left[\sum_{j=\ell+1}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]\Big|\mathscr{F}(s)\right] = 0$$

proof :

The fourth term

$$④\ \mathbb{E}\left\{\Delta(t_k)\big(W(t)-W(t_k)\big)\,\Big|\,\mathscr{F}(s)\right\} = \mathbb{E}\left\{\mathbb{E}\left[\Delta(t_k)\big(W(t)-W(t_k)\big)\,\Big|\,\mathscr{F}(t_k)\right]\Big|\,\mathscr{F}(s)\right\}$$

$$= \mathbb{E}\left\{\Delta(t_k)\left(\mathbb{E}\left[W(t)\,\Big|\,\mathscr{F}(t_k)\right]-W(t_k)\right)\Big|\,\mathscr{F}(s)\right\}$$

$$= \mathbb{E}\left\{\Delta(t_k)\big(W(t_k)-W(t_k)\big)\,\Big|\,\mathscr{F}(s)\right\} = 0$$

$$\mathbb{E}\left[I(t)\,\Big|\,\mathscr{F}(s)\right] = \sum_{j=0}^{\ell-1}\Delta(t_j)[W(t_{j+1})-W(t_j)] + \Delta(t_\ell)[W(t_{\ell+1})-W(t_\ell)] + \sum_{j=\ell+1}^{k-1}\Delta(t_j)[W(t_{j+1})-W(t_j)] + \Delta(t_k)[W(t)-W(t_k)]$$

$$= \sum_{j=0}^{\ell-1}\Delta(t_j)[W(t_{j+1})-W(t_j)] + \Delta(t_\ell)\big(W(s)-W(t_\ell)\big) = I(s)$$

$proof:$

Because $I(t)$ is a martingale and $I(0) = 0$,

$$\mathbb{E}\left[I(t)\middle|\mathscr{F}(s)\right] = I_0 \Rightarrow \mathbb{E}\left\{\mathbb{E}\left[I(t)\middle|\mathscr{F}(0)\right]\right\} = \mathbb{E}(I_0)$$

$$\Rightarrow \mathbb{E}I(t) = \mathbb{E}(I_0) = \mathbb{E}(0) = 0$$

$$\Rightarrow Var\, I(t) = \mathbb{E}\, I^2(t) - \left[\mathbb{E}\, I(t)\right]^2 = \mathbb{E}\, I^2(t)$$

Theorem 4.2.2. ***(Itô isometry). The Itô integral defined by (4.2.2) satisfies***

$$\mathbb{E}\, I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\, du$$

proof :

Let $D_j = W(t_{j+1}) - W(t_j)$ *for* $j = 0, \ldots, k-1$ *and* $D_k = W(t) - W(t_k)$

so that $I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] = \sum_{j=0}^{k} \Delta(t_j)\, D_j$

$$I^2(t) = \left(\sum_{j=0}^{k} \Delta(t_j)\, D_j\right)^2 = \underbrace{\sum_{j=0}^{k} \Delta^2(t_j)\, D_j^2}_{①} + \underbrace{2\sum_{0\le i<j\le k} \Delta(t_i)\, \Delta(t_j)\, D_i\, D_j}_{②}$$

proof :

show that the expected value of each of the cross terms is zero.For $i < j$, the random variable $\Delta(t_i)\Delta(t_j)D_i$ is $\mathscr{F}(t_j) - measurable$, while the Brownian increment D_j is independent of $\mathscr{F}(t_j)$. Furthermore,$\mathbb{E}D_j = 0$. Therefore

$$\mathbb{E}\left[\Delta(t_i)\Delta(t_j)D_iD_j\right] = \mathbb{E}\left[\Delta(t_i)\Delta(t_j)D_i\right] \cdot \mathbb{E}D_j = \mathbb{E}\left[\Delta(t_i)\Delta(t_j)D_i\right] \cdot 0 = 0.$$

※ D_j is independent of $\mathscr{F}(t_j)$

$D_j = W(t_{j+1}) - W(t_j)$

$\Rightarrow D_j$ 不是 $\mathscr{F}(t_j) - measurable$

$\Rightarrow D_j$ is independent of $\mathscr{F}(t_j)$

※ $\mathbb{E}D_j = 0$

D_j : 布朗運動的增量

$D_j = W(t_{j+1}) - W(t_j) \sim N(0, t_{j+1} - t_j)$

$\mathbb{E}D_j = 0, Var(D_j) = \mathbb{E}D_j^2 = t_{j+1} - t_j$

$proof:$

consider the square terms $\Delta^2(t_j)\,D_j^2$.The random variable $\Delta^2(t_j)$ is $\mathscr{F}(t_j) - measurable$, and the squared Brownian increment D_j^2 is independent of $\mathscr{F}(t_j)$.Furthermore, $\mathbb{E}D_j^2 = t_{j+1} - t_j$ for $j = 0,...,k-1$ and $\mathbb{E}D_k^2 = t - t_k$. Therefore,

$$\mathbb{E}I^2(t) = \sum_{j=0}^{k} \mathbb{E}\left[\Delta^2(t_j)\,D_j^2\right] = \sum_{j=1}^{k} \mathbb{E}\Delta^2(t_j)\cdot\mathbb{E}D_j^2$$

$$= \sum_{j=1}^{k-1} \mathbb{E}\Delta^2(t_j)\,(t_{j+1} - t_j) + \mathbb{E}\Delta^2(t_k)(t - t_k)\,.$$

proof :

But $\Delta(t_j)$ is constant on the interval $[t_j, t_{j+1})$, and hence $\Delta^2(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \Delta^2(u)\, du$.

Similarly, $\Delta^2(t_k)(t - t_k) = \int_{t_k}^{t} \Delta^2(u)\, du$.

$$\mathbb{E}I^2(t) = \sum_{j=0}^{k} \mathbb{E}\left[\Delta^2(t_j)\right] \times \mathbb{E}D_j^2 = \sum_{j=0}^{k-1} \mathbb{E}\left[\Delta^2(t_j)\right] \times (t_{j+1} - t_j) + \mathbb{E}\left[\Delta^2(t_k)\right] \times (t - t_k)$$

$$= \sum_{j=0}^{k-1} \mathbb{E}\left[\int_{t_j}^{t_{j+1}} \Delta^2(u)\, du\,.\right] + \mathbb{E}\left[\int_{t_k}^{t} \Delta^2(u)\, du\,.\right] = \mathbb{E}\left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Delta^2(u)\, du. + \int_{t_k}^{t} \Delta^2(u)\, du\,.\right] = \mathbb{E}\int_0^t \Delta^2(u)\, du\,.$$

Theorem 4.2.3. ***The quadratic variation accumulated up to time by the Itô integral***

$$[I,I](t) = \int_0^t \Delta^2(u)\,du\,.$$

Recall : ***Theorem3.4.3*** Let W be a Brownian motion. Then $[W,W](T) = T$ for all $T \geq 0$

proof :

We first compute the quadratic variation accumulated by the Itô integral on one of the subintervals $[t_j, t_{j+1}]$ on which $\Delta(u)$ is constant. Choose partition points

$t_j = s_0 < s_1 < , \ldots, < s_m = t_{j+1}$, and consider

$$\sum_{i=0}^{m-1} \left[I(s_{i+1}) - I(s_i)\right]^2 = \sum_{i=0}^{m-1} \left[\Delta(t_j)\big(W(s_{i+1}) - W(s_i)\big)\right]^2 = \Delta^2(t_j) \sum_{i=0}^{m-1} \big(W(s_{i+1}) - W(s_i)\big)^2$$

※ $I(s_{i+1}) - I(s_i) = \int_{s_i}^{s_{i+1}} \Delta(t_j) dW(u) = \Delta(t_j)[W(s_{i+1} - W(s_i)]$ 布朗運動的增量性質

$proof:$

As $m \to \infty$ and the step size $max_{i=0,\ldots,m-1}(s_{i+1} - s_i)$ approaches zero

$$\sum_{i=0}^{m-1} \left[\left(W(s_{i+1}) - W(s_i) \right) \right]^2 \to \sum_{i=0}^{m-1} \left(s_{i+1} - s_i \right) = t_{j+1} - t_j$$

$$\Delta^2(t_j) \sum_{i=0}^{m-1} \left(W(s_{i+1}) - W(s_i) \right)^2 = \Delta^2(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \Delta^2(u)du$$

We can get the quadratic variation accumulated by the Itô integral on intervals $[0,t]$

$$[I, I](t) = \int_0^t \Delta^2(u)\, du\,.$$

Quadratic variation:

1. Computed path-by-path, the result can depend on the path.

2. Depends on the size of the positions we take

   We choose large positions $\Delta(u)$, the Itô integral will have a large quadratic variation.

   Choose small positions $\Delta(u)$ and the Itô integral would have a small quadratic variation.

3. Regarded as a measure of risk.

The variance of I(t)

1. Average over all possible paths of the quadratic variation.

2. Cannot be random.

3. More theoretical concept than quadratic variation.

Empirical variance

1. Computed from a realized path and is an estimator of the theoretical variance.

Remark 4.2.4. (on notation). The notations

$I(t) = \int_0^t \Delta(u)\,dW(u)$ (4.2.11) $I(t) = I(0) + \int_0^t \Delta(u)\,dW(u)$ (4.2.13)	**integral form**
$dI(t) = \Delta(t)dW(t)$ (4.2.12)	**differential form**

Recall : $dW(t)\,dW(t) = dt \Longleftrightarrow [W,W](t) = t, t \geq 0$

$$dI(t)\,dI(t) = \Delta(t)dW(t) \times \Delta(t)dW(t) = \Delta^2(t)[dW(t)]^2 = \Delta^2(t)dt$$

$dI(t)\,dI(t) = [I,I](t) = \Delta^2(t)dt$ is another way of the result of Theorem 4.2.3

※ **Theorem4.2.3** $[I,I](t) = \int_0^t \Delta^2(u)\,du$