Stochastic Calculus For Finance - volume 2

Section 8.2 Stopping times

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Introduction

The goal of this section is to define a concept, stopping times, to discuss further properties or theorems for American-style derivatives. We will walk through some definitions, examples and an important theorem-Stopped Martingale Theorem.

Recall - Definition for discrete case

In an N-period model, a **stopping time** (or **optional time**) is a random variable τ such that $\Omega \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ and satisfies the followings:

$$\tau(w_1, w_2, \cdots, w_n, w_{n+1}, \cdots, w_N) = \tau(w_1, w_2, \cdots, w_n, w'_{n+1}, \cdots, w'_N) = n$$

for all $w_i \in \Omega, i = 1, 2, \cdots$

Examples

Suppose you are gambling on a coin flipping process, where you start at \$10 and loses \$1 for every flip on tail, and win \$1 for every flip on head. An example of a stopping time is the process that your wealth first reach \$15 or \$0. For example, if your wealth first hit \$15 at the seventh toss. Then τ=7, no matter what happens after the seventh toss.

An example that is not a stopping time is the process that your wealth reaches the maximum of the whole process, which can not be determined at some specific time except at the end of the process.

A quick way to distinguish whether a random variable is a stopping time is if you are certain if the process reaches a particular behavior at a given time and all its information aforehand.

Definition for continuous case

A **stopping time** is a random variable τ such that $\tau : \Omega \to [0, \infty]$ and satisfies:

$$\{\tau \leq t\} \in F_t \ for \ all \ t \geq 0$$

where F_t is the filtration up to time t.

Remark

Remark 8.2.2. Let $t \geq 0$ be given. Note that (8.2.1) and the properties of σ-algebras imply that $\{\tau > t - \frac{1}{n}\} = \{\tau \leq t - \frac{1}{n}\}^c \in \mathcal{F}\left(t - \frac{1}{n}\right)$ for all positive integers n. Since every set in $\mathcal{F}\left(t - \frac{1}{n}\right)$ is also in $\mathcal{F}(t)$, we conclude that $\{\tau > t - \frac{1}{n}\}$ is in $\mathcal{F}(t)$ for every n, and hence

$$\{\tau = t\} = \{\tau \leq t\} \cap \left(\bigcap_{n=1}^{\infty} \left\{\tau > t - \frac{1}{n}\right\}\right)$$

is also in $\mathcal{F}(t)$. In other words, by Definition 8.2.1, a stopping time τ has the property that the decision to stop at time t must be based on information available at time t.

Example-First passage time

Let X(t) be an adapted process with continuous paths. We define:

$\tau_m = min\{t \geq 0, X(t) = m\}, for\ m\ be\ a\ number$, which is the first time X(t) reaches level m. If X never reaches level m, then we define $\tau_m = \infty$, obviously, τ_m is a stopping time since the value can be determined by the path up to τ_m.

We verify this fact by using the definition for stopping time for continuous case. Without loss of generosity, we consider the case that X does reach level m at some time t.

Example continue-First passage time

We want to show that $\tau_m = min\{t \geq 0, X(t) = m\}, for\ m\ be\ a\ number$, satisfies the definition of stopping time, i.e., $\{\tau \leq t\} \in F_t\ for\ all\ t \geq 0$.

For t = 0, we have $\{\tau \leq t\} = \{\tau = 0\}$, which is either $\Omega\ or\ \phi$ depending on whether X(0) is equal to m or not, hence $\{\tau = 0\} \in F_0$.

Example continue-First passage time

For t > 0, we define the running maximum of X on [0,t], which is: $M(t) := \max_{s\in[0,t]} X(s)$

We claim that $\{\tau_m \le t\} = \{M(t) \ge m\}$, if the claim is proven, then since M(t) is a F_t-measurable function, then $\{\tau_m \le t\} = \{M(t) \ge m\} \in F_t$ and the proof is done.

To prove the claim, first suppose that $\tau_m \le t$ for some t, which means $\exists s \in [0,t]\ such\ that\ X(s) = m$

But this also means that $M(t) := \max_{s\in[0,t]} X(s) \ge m$, and hence we have $\{\tau_m \le t\} \subseteq \{M(t) \ge m\}$.

Next suppose that $M(t) \ge m$ for some real m, i.e., $\max_{s\in[0,t]} X(s) \ge m$. Since X is continuous, by the Intermediate Value Theorem, $\exists s \in [0,t]\ such\ that\ X(s) = m$. By picking the smallest s, we have that $\tau_m = s \le t$, and hence $\{M(t) \ge m\} \subseteq \{\tau_m \le t\}$, hence we have $\{\tau_m \le t\} = \{M(t) \ge m\}$.

Definition-Stopped process

Let X(t) be an adapted process and τ be a stopping time, then a stopped process is defined as $\widetilde{X}(t) = X(t \wedge \tau) = X(min\{t, \tau\})$.

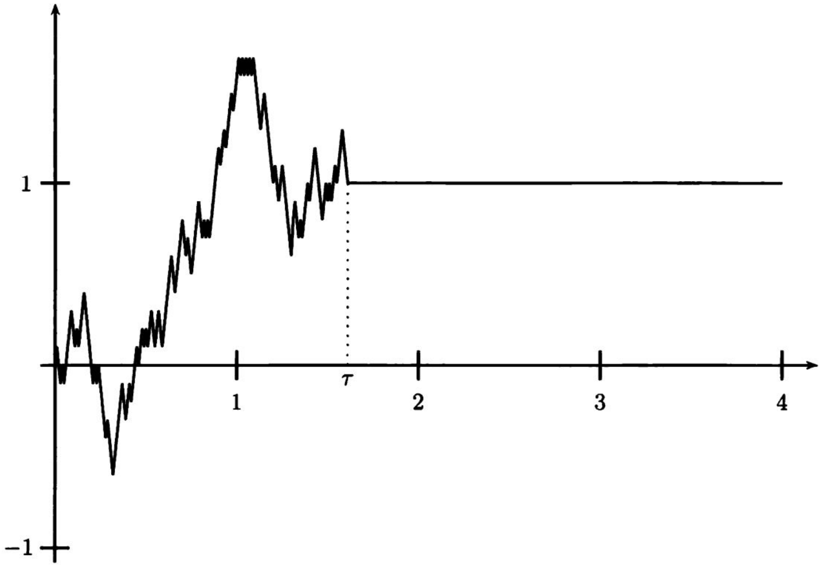

Fig. 8.2.1. A stopped process.

Theorem-Stopped Martingale

If (X(t), t ≥ 0) is a martingale(supermartingale; submartingale), then the stopped process $\widetilde{X}(t) = X(t \wedge \tau) = X(min\,\{t, \tau\})$ is also a martingale(supermartingale;submartingale), for some stopping time τ .

Note:

(ii) If

$$\mathbb{E}[M(t)|\mathcal{F}(s)] \geq M(s) \text{ for all } 0 \leq s \leq t \leq T,$$

we say this process is a submartingale. *It has no tendency to fall; it may have a tendency to rise.*

(iii) If

$$\mathbb{E}[M(t)|\mathcal{F}(s)] \leq M(s) \text{ for all } 0 \leq s \leq t \leq T,$$

we say this process is a supermartingale. *It has no tendency to rise; it may have a tendency to fall.*