Stochastic Calculus For Finance - volume 2

Section 7.2 Maximum of Brownian Motion with Drift

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Introduction

- The goal of this section is to derive the joint density for a Brownian motion with drift and its maximum to date.
- Given a Brownian motion $\widetilde{W(t)}, 0 \leq t \leq T$ on a probability space $(\Omega, F, \widetilde{\mathbb{P}})$
- Assume that $\widetilde{W(t)}$ is a martingale (no drift) under $\widetilde{\mathbb{P}}$, and starts at 0
- We else define a Brownian motion $\widehat{W(t)} = at + \widetilde{W(t)}, 0 \leq t \leq T, a \in \mathbb{R}$, having a drift under $\widetilde{\mathbb{P}}$
- We then define $\widehat{M(t)} = \max_{0 \leq t \leq T} \widehat{W(t)}$
- We observe that $\widehat{W(0)} = 0$ so $\widehat{M(t)} \geq 0$,

  and also clearly we have $\widehat{M(t)} \geq \widehat{W(t)}$

- We would like to discuss the pair of

  random variables $(\widehat{M(t)}, \widehat{W(t)})$

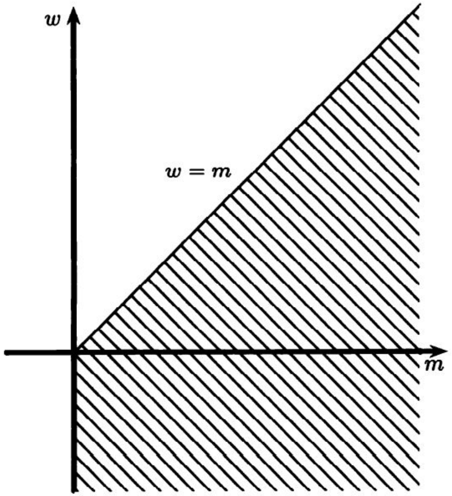

Fig. 7.2.1. Range of $(\widehat{M}(T), \widehat{W}(T))$.

Theorem 7.2.1. *The joint density under* $\widetilde{\mathbb{P}}$ *of the pair* $\big(\widehat{M}(T), \widehat{W}(T)\big)$ *is*

$$\widetilde{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, \quad w \le m,\ m \ge 0, \tag{7.2.3}$$

and is zero for other values of m *and* w.

recall:

Theorem 3.7.3. *For $t > 0$, the joint density of $(M(t), W(t))$ is*

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \le m,\ m > 0. \tag{3.7.7}$$

Theorem 5.2.3 (Girsanov, one dimension). *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\, dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\, du\right\}, \tag{5.2.11}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du, \tag{5.2.12}$$

and assume that[1]

$$\mathbb{E}\int_0^T \Theta^2(u) Z^2(u)\, du < \infty. \tag{5.2.13}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by (5.2.1), the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, dP(\omega) \text{ for all } A \in \mathcal{F}.$$

proof:

We want to show that $\widehat{W(t)}$ is a Brownian motion with no drift at a new probability measure $\widehat{\mathbb{P}}$ defined by the exponential martingale $\widehat{Z}(t) = e^{-\alpha\widetilde{W}(t)-\frac{1}{2}\alpha^2 t} = e^{-\alpha\widehat{W}(t)+\frac{1}{2}\alpha^2 t}, \quad 0 \le t \le T,$

Using the notation of Girsanov Theorem, we take $\theta(u) = a$, hence we have :

$\int_0^t \theta(u) d\widetilde{W(t)} = \int_0^t a d\widetilde{W(t)} = a(\widetilde{W(t)} - \widetilde{W(0)}) = a\widetilde{W(t)}$, and $\int_0^t \theta^2(u)du = \int_0^t a^2 du = a^2 t$

By Girsanov Theorem, $Z(t) = \exp\left\{-\int_0^t \Theta(u)\, dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\, du\right\} = e^{-\alpha\widetilde{W}(t)-\frac{1}{2}\alpha^2 t}$

And $\widehat{W(t)} = \widetilde{W(t)} + \int_0^t a du = at + \widetilde{W(t)}, 0 \le t \le T, a \in \mathbb{R}$, hence $\widehat{W(t)}$ is a Brownian motion with zero drift under the probability measure $\widehat{\mathbb{P}}$, since we know that $\widehat{W(t)}$ is a martingale under $\widehat{\mathbb{P}}$

Now by Theorem 3.7.3 we have the joint density for $(\widehat{M(t)}, \widehat{W(t)})$ under $\widehat{\mathbb{P}}$ is

$$\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2m-w)^2}, \quad w \le m, \; m \ge 0, \qquad (7.2.4)$$

proof(continue):

To obtain the joint density of $(\widehat{M(t)}, \widehat{W(t)})$ under $\widetilde{\mathbb{P}}$, we recall Lemma 5.2.1,

Lemma 5.2.1. *Let t satisfying $0 \le t \le T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]. \tag{5.2.8}$$

And hence we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le w\} \\
&= \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\right] \\
&= \widehat{\mathbb{E}}\left[\frac{1}{\widehat{Z}(T)}\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\right] \\
&= \widehat{\mathbb{E}}\left[e^{\alpha\widehat{W}(T)-\frac{1}{2}\alpha^2 T}\mathbb{I}_{\{\widehat{M}(T)\le m, \widehat{W}(T)\le w\}}\right] \\
&= \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y-\frac{1}{2}\alpha^2 T}\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(x,y)\,dx\,dy.
\end{aligned}$$

Therefore, the density of $(\widehat{M}(T), \widehat{W}(T))$ under $\widetilde{\mathbb{P}}$ is

$$\frac{\partial^2}{\partial m\,\partial w}\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le w\} = e^{\alpha w-\frac{1}{2}\alpha^2 T}\widehat{f}_{\widehat{M}(T),\widehat{W}(T)}(m,w). \tag{7.2.5}$$

Corollary 7.2.2. *We have*

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \leq m\} = N\left(\frac{m - \alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), \quad m \geq 0, \quad (7.2.6)$$

and the density under $\widetilde{\mathbb{P}}$ *of the random variable* $\widehat{M}(T)$ *is*

$$\widetilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), \; m \geq 0, \quad (7.2.7)$$

and is zero for $m < 0$.

proof:

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\}$$

$$= \int_0^m \int_w^m \frac{2(2\mu - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2}\, d\mu\, dw$$

$$+ \int_{-\infty}^0 \int_0^m \frac{2(2\mu - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2}\, d\mu\, dw$$

$$= -\int_0^m \frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2} \Big|_{\mu=w}^{\mu=m} dw$$

$$- \int_{-\infty}^0 \frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\mu - w)^2} \Big|_{\mu=0}^{\mu=m} dw$$

$$= -\frac{1}{\sqrt{2\pi T}} \int_0^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m - w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_0^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2} dw$$

$$- \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^0 e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m - w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^0 e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2} d$$

$$= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m - w)^2} dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2} dw.$$

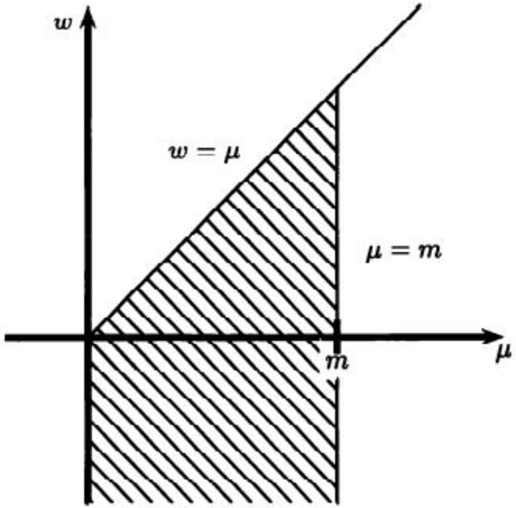

Fig. 7.2.2. The region $\widehat{M}(T) \le m$.

proof(continue):

We complete the squares. Observe that

$$-\frac{1}{2T}(w-2m-\alpha T)^2 = -\frac{(2m-w)^2}{2T} + \alpha w - 2\alpha m - \frac{1}{2}\alpha^2 T,$$

$$-\frac{1}{2T}(w-\alpha T)^2 = -\frac{w^2}{2T} + \alpha w - \frac{1}{2}\alpha^2 T.$$

Therefore,

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\}$$
$$= -\frac{e^{2\alpha m}}{\sqrt{2\pi T}} \int_{-\infty}^{m} e^{-\frac{1}{2T}(w-2m-\alpha T)^2}\, dw + \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{m} e^{-\frac{1}{2T}(w-\alpha T)^2}\, dw.$$

We make the change of variable $y = \frac{w-2m-\alpha T}{\sqrt{T}}$ in the first integral and $y = \frac{w-\alpha T}{\sqrt{T}}$ in the second, thereby obtaining

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} = -\frac{e^{2\alpha m}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\, dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\, du$$
$$= -e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + N\left(\frac{m-\alpha T}{\sqrt{T}}\right).$$

proof(continue):

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\} = -\frac{e^{2\alpha m}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\,dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\,du$$
$$= -e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + N\left(\frac{m-\alpha T}{\sqrt{T}}\right).$$

To obtain the density (7.2.7), we differentiate (7.2.6) with respect to m:

$$\frac{d}{dm}\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m\}$$
$$= N'\left(\frac{m-\alpha T}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{T}}\right) - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right)$$
$$-e^{2\alpha m} N'\left(\frac{-m-\alpha T}{\sqrt{T}}\right)\left(-\frac{1}{\sqrt{T}}\right)$$
$$= \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + \frac{e^{2\alpha m}}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(-m-\alpha T)^2}.$$

The exponent in the third term is

$$\begin{aligned} 2\alpha m - \frac{(-m-\alpha T)^2}{2T} &= \frac{4\alpha m}{2T} - \frac{m^2 + 2\alpha m T + \alpha^2 T^2}{2T} \\ &= -\frac{m^2 - 2\alpha m T + \alpha^2 T^2}{2T} \\ &= -\frac{(m-\alpha T)^2}{2T}, \end{aligned}$$

which is the exponent in the first term. Combining the first and third terms, we obtain (7.2.7). □