# Stochastic Calculus For Finance - volume 2
# Section 5.4 Fundamental Theorems of Assets Pricing

06/03/2025,柯宥寧

# Terminology

- W(t) = (W1(t), W2(t), … , Wd(t)), is a d-dimensional Brownian motion on the actual probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the actual probability measure.

- Associating to this Brownian motion, we have the filtration (multidimensional) F(t), also we have a fixed final time T, and we denote F(T) as F.

# Theorem 5.4.1 (Girsanov Theorem for Multidimensional)

**Theorem 5.4.1 (Girsanov, multiple dimensions).** *Let $T$ be a fixed positive time, and let $\Theta(t) = \big(\Theta_1(t), \ldots, \Theta_d(t)\big)$ be a d-dimensional adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right\}, \qquad (5.4.1)$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du, \qquad (5.4.2)$$

*and assume that*

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)\, du < \infty. \qquad (5.4.3)$$

*Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$, and under the probability measure $\widetilde{\mathbb{P}}$ given by*

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

*the process $\widetilde{W}(t)$ is a d-dimensional Brownian motion.*

*basic concept:*

- Multidimensional Ito's integral can be viewed as a matrix form integral: $\int_0^T \Theta dW = \int_0^T \begin{pmatrix} \Theta_{11} & \cdots & \Theta_{1d} \\ \vdots & \ddots & \vdots \\ \Theta_{m1} & \cdots & \Theta_{md} \end{pmatrix} d \begin{pmatrix} W_1 \\ \vdots \\ W_d \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^d \int_0^T \Theta_{1j} dW_j \\ \vdots \\ \sum_{j=1}^d \int_0^T \Theta_{mj} dW_j \end{pmatrix}$, and hence $\int_0^t \Theta(u) \cdot dW(u) = \sum_{j=1}^d \int_0^t \Theta_j(u) dW_j(u)$
- $\|\Theta(u)\|$ denotes the Euclidean norm, which is $\|\Theta(u)\| = ((\sum_{j=1}^d \Theta_j^2(u)))^{\frac{1}{2}}$
- $\widetilde{W} = W(t) + \int_0^t \Theta(u) du$ simply represents $\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u) du,\ j = 1, ..., d$
- The condition $\mathbb{E}(\int_0^T \|\Theta(u)\|^2 Z^2(u) du) < \infty$ ensures that the defining Ito's integral is well-defined and is a martingale
- Moreover, it ensures the process is in $L^2$ , and ensures the Hilbert properties

## *recall:* Multidimensional Levy Theorem

- Let W1(t), W2(t), …, Wd(t), t $\geq$0, be martingales relative to a filtration F(t).
- All Wi(0)=0, i=1,2, …,d.
- All Wi(t) has continuous paths.
- All [Wi,Wi](t)=t, for all t $\geq$0.
- All [Wi,Wj](t)=0, for all i $\neq$j, and i,j=1,2, …,d.

Then we have W1,W2, ..., Wd are independent Brownian motions. Or we can also see that let $W(t) = (W_1(t), ..., W_d(t))$, then W is a standard d-dimensional Brownian motion, and we would like to specifically recall the fact that by definition, $dW_i(t)dW_j(t) = 0, \forall i \neq j$

*proof:*

We want to show that $\widetilde{W}$ satisfies the Multidimensional Levy's Characterization Theorem and hence we can conclude that $\widetilde{W}$ is a standard d-dimensional Brownian motion.

Before that, we would first prove:

$Z(t) = exp(-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 \, du)$ , then $\mathbb{E}(Z(t)) = 1, \forall t \geq 0$

Simply set $f(x) = e^x, X(t) = -\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 \, du$ , then we have

$dZ(t) = d(f(X(t))) = f'(X(t))dX + \frac{1}{2}f''(X(t))(dX)^2$

$= e^{X(t)}(-\sum_{j=1}^{d}\Theta_j(u)dW_j(u) - \frac{1}{2}\|\Theta(u)\|^2 \, du) + \frac{1}{2}e^{X(t)}\|\Theta(u)\|^2 \, du = -\sum_{j=1}^{d}\Theta_j(u)Z(t)dW_j(u)$, hence martingale.

Therefore, $\mathbb{E}(Z(t)) = \mathbb{E}(\mathbb{E}(Z(t)|F(0))) = \mathbb{E}(1) = 1, \forall t \geq 0$

Now we prove the rest:

*continue:*

Easily observe that all $W_j(u)$ start at 0, having continuous paths, also the quadratic variation can be computed as $\widetilde{dW(t)}^2 = (dW(t) + \Theta(t)dt)^2 = dt$

Also we see that $\widetilde{dW_i(t)}\widetilde{dW_j(t)} = (dW_i(t) + \Theta_i(t)dt)(dW_j(t) + \Theta_j(t)dt) = 0.\forall i \neq j$

To prove that $\widetilde{W(t)}$ is a martingale under $\widetilde{\mathbb{P}}$, consider:

$$d(\widetilde{W(t)}Z(t)) = \widetilde{W(t)}dZ(t) + d\widetilde{W(t)}Z(t) + d\widetilde{W(t)}dZ(t) =$$

$$\widetilde{W(t)}(-\sum_{j=1}^{d}\Theta_j(t)Z(t)dW_j(u)) + Z(t)(dW(t)+\Theta(t)dt) + (dW(t)+\Theta(t)dt)(-\sum_{j=1}^{d}\Theta_j(t)Z(t)dW_j(t)) = (-\widetilde{W(t)}\Theta(t)+1)Z(t)dW(t)$$

Hence, using Lemma 5.2.2, we obtain $\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\widetilde{W}(s)Z(s) = \widetilde{W}(s)$,

and the proof is done.

**Lemma 5.2.2.** *Let $s$ and $t$ satisfying $0 \leq s \leq t \leq T$ be given and let $Y$ be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]. \qquad (5.2.9)$$

# Theorem 5.4.2 (Multidimensional Martingale Representation )

*Let $T$ be a fixed positive time, and assume that $\mathcal{F}(t)$, $0 \leq t \leq T$, is the filtration generated by the d-dimensional Brownian motion $W(t)$, $0 \leq t \leq T$. Let $M(t)$, $0 \leq t \leq T$, be a martingale with respect to this filtration under $\mathbb{P}$. Then there is an adapted, d-dimensional process $\Gamma(u) = (\Gamma_1(u), \ldots, \Gamma_d(u))$, $0 \leq u \leq T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u), \; 0 \leq t \leq T. \tag{5.4.4}$$

*If, in addition, we assume the notation and assumptions of Theorem 5.4.1 and if $\widetilde{M}(t)$, $0 \leq t \leq T$, is a $\widetilde{\mathbb{P}}$-martingale, then there is an adapted, d-dimensional process $\widetilde{\Gamma}(u) = (\widetilde{\Gamma}_1(u), \ldots, \widetilde{\Gamma}_d(u))$ such that*

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \cdot d\widetilde{W}(u), \; 0 \leq t \leq T. \tag{5.4.5}$$

*proof(simple):*

We first introduce a important theorem,Ito’s Representation theorem:

$$\textit{Let } F \in L^2(\mathcal{F}^n(t), \mathbb{P}),\ \textit{then there uniquely exists a stochastic process } \Theta(t,w), \textit{ which is progressively measurable such that } F(w) = \mathbb{E}(F) + \int_0^T \Theta(t,w)\,dW(t)$$

Now to prove our case. simply take F=M(t) and we are done.

# Multidimensional Market model

Assuming there are m stocks, each having SDE $dS_i(t) = \alpha_i(t)S_i(t)\,dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t), \quad i = 1, \ldots, m.$

It is clear that these stocks are correlated, to see this we define:

$\sigma_i(t) = \sqrt{\sum_{j=1}^{d}\sigma_{ij}^2(t)} \neq 0$ and $B_i(t) = \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}dW_j(u), i = 1, ..., m$

We see that each B is a continuous martingale, and $(dB_i(t))^2 = \sum_{j=1}^{d}\frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}dt = dt$

By Levy's Theorem, each B is a Brownian motion, then we rewrite:

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + \sigma_i(t)S_i(t)\,dB_i(t).$$

We see that $dB_i(t)dB_k(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}dt = \rho_{ik}(t)dt, \forall i \neq k$, using product rules:

$$d(B_i(t)B_k(t)) = B_i(t)dB_k(t) + B_k(t)dB_i(t) + dB_i(t)B_k(t)$$

*Continue:*

We can obtain: $B_i(t)B_k(t) = \int_0^t B_i(u)dB_k(u) + \int_0^t B_k(u)dB_i(u) + \int_0^t \rho_{ik}(u)du$

Taking expectation on both sides: $Cov[B_i(t), B_k(t)] = \mathbb{E}(\int_0^t \rho_{ik}(u)du)$

If $\sigma_{ij}(t), \sigma_{kj}(t)\ is\ a\ constant,\ then\ Cov[B_i(t), B_k(t)] = \rho t,\ since B_i(t), B_k(t)\ both\ have\ variance\ t,$ $so\ the\ correlation\ is\ \rho,\ which\ we\ call\ it\ instantaneous\ correlation.$

We note that: $dS_i(t)dS_k(t) = \sigma_i(t)\sigma_k(t)S_i(t)S_k(t)dB_i(t)dB_k(t) = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)S_i(t)S_k(t)dt$

or $\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_k(t)}{S_k(t)} = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)dt$

Finally we define a *discount process:* $D(t) = e^{-\int_0^t R(u)du}$, assuming that R is an adapted process, then we have:

$$d(D(t)S_i(t)) = D(t)dS_i(t) - R(t)S_i(t)dt = D(t)S_i(t)[(\alpha_i(t) - R(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]$$

$$= D(t)S_i(t)[(\alpha_i(t) - R(t)dt) + \sigma_i(t)dB_i(t)],\ i = 1, ..., m$$