Stochastic Calculus For Finance - volume 2
Section 4.6 Multivariable Stochastic Calculus

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Definition

4.6.1 Multiple Brownian Motions

Definition 4.6.1. *A* d-dimensional Brownian motion *is a process*

$$W(t) = (W_1(t), \ldots, W_d(t))$$

with the following properties.

(i) *Each* $W_i(t)$ *is a one-dimensional Brownian motion.*
(ii) *If* $i \neq j$, *then the processes* $W_i(t)$ *and* $W_j(t)$ *are independent.*

Associated with a d-dimensional Brownian motion, we have a filtration $\mathcal{F}(t)$, $t \geq 0$, *such that the following holds.*

(iii) **(Information accumulates)** *For* $0 \leq s < t$, *every set in* $\mathcal{F}(s)$ *is also in* $\mathcal{F}(t)$.
(iv) **(Adaptivity)** *For each* $t \geq 0$, *the random vector* $W(t)$ *is* $\mathcal{F}(t)$*-measurable.*
(v) **(Independence of future increments)** *For* $0 \leq t < u$, *the vector of increments* $W(u) - W(t)$ *is independent of* $\mathcal{F}(t)$.

Notes

$$dW_i(t)\,dW_i(t) = dt. \; dW_i(t)\,dW_j(t) = 0, \quad i \neq j.$$

proof:

Let $\Pi = \{t_0, \dots, t_n\}$ be a partition of $[0, T]$. For $i \neq j$, define the *sampled cross variation* of W_i and W_j on $[0, T]$ to be

$$C_\Pi = \sum_{k=0}^{n-1} \big[W_i(t_{k+1}) - W_i(t_k)\big]\big[W_j(t_{k+1}) - W_j(t_k)\big].$$

The increments appearing on the right-hand side of the equation above are all independent of one another and all have mean zero. Therefore, $\mathbb{E}C_\Pi = 0$.

proof(continue):

$$C_\Pi = \sum_{k=0}^{n-1} \left[W_i(t_{k+1}) - W_i(t_k)\right]\left[W_j(t_{k+1}) - W_j(t_k)\right].$$

We compute $\mathrm{Var}(C_\Pi)$. Note first that

$$C_\Pi^2 = \sum_{k=0}^{n-1} \left[W_i(t_{k+1}) - W_i(t_k)\right]^2 \left[W_j(t_{k+1}) - W_j(t_k)\right]^2$$
$$+2\sum_{\ell<k}^{n-1} \left[W_i(t_{\ell+1}) - W_i(t_\ell)\right]\left[W_j(t_{\ell+1}) - W_j(t_\ell)\right]$$
$$\cdot\left[W_i(t_{k+1}) - W_i(t_k)\right]\left[W_j(t_{k+1}) - W_j(t_k)\right].$$

All the increments appearing in the sum of cross-terms are independent of one another and all have mean zero. Therefore,

$$\mathrm{Var}(C_\Pi) = \mathbb{E}C_\Pi^2 = \mathbb{E}\sum_{k=0}^{n-1} \left[W_i(t_{k+1}) - W_i(t_k)\right]^2 \left[W_j(t_{k+1}) - W_j(t_k)\right]^2.$$

But $\left[W_i(t_{k+1}) - W_i(t_k)\right]^2$ and $\left[W_j(t_{k+1}) - W_j(t_k)\right]^2$ are independent of one another, and each has expectation $(t_{k+1} - t_k)$. It follows that

$$\mathrm{Var}(C_\Pi) = \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \le \|\Pi\| \cdot \sum_{k=0}^{n-1} (t_{k+1} - t_k) = \|\Pi\| \cdot T.$$

As $\|\Pi\| \to 0$, we have $\mathrm{Var}(C_\Pi) \to 0$, so C_Π converges to the constant $\mathbb{E}C_\Pi = 0$.

Ito-Doeblin formula for Multiple processes(for 2-dimentional)

Let $X(t)$ and $Y(t)$ be Itô processes, which means they are processes of the form

$$X(t) = X(0) + \int_0^t \Theta_1(u)\,du + \int_0^t \sigma_{11}(u)\,dW_1(u) + \int_0^t \sigma_{12}(u)\,dW_2(u),$$
$$Y(t) = Y(0) + \int_0^t \Theta_2(u)\,du + \int_0^t \sigma_{21}(u)\,dW_1(u) + \int_0^t \sigma_{22}(u)\,dW_2(u).$$

The integrands $\Theta_i(u)$ and $\sigma_{ij}(u)$ are assumed to be adapted processes. In differential notation, we write

$$dX(t) = \Theta_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) + \sigma_{12}(t)\,dW_2(t), \tag{4.6.1}$$
$$dY(t) = \Theta_2(t)\,dt + \sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t). \tag{4.6.2}$$

The Itô integral $\int_0^t \sigma_{11}(u)\,dW_1(u)$ accumulates quadratic variation at rate $\sigma_{11}^2(t)$ per unit time, and the Itô integral $\int_0^t \sigma_{12}(u)\,dW_2(u)$ accumulates quadratic variation at rate $\sigma_{12}^2(t)$ per unit time. Because both of these integrals appear in $X(t)$, the process $X(t)$ accumulates quadratic variation at rate $\sigma_{11}^2(t) + \sigma_{12}^2(t)$ per unit time:

$$[X, X](t) = \int_0^t \left(\sigma_{11}^2(u) + \sigma_{12}^2(u)\right) du.$$

We may write this equation in differential form as

$$dX(t)\,dX(t) = \left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) dt. \tag{4.6.3}$$

proof:

$$dX(t)\,dX(t) = \left(\theta_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t)\right)^2$$

$$= \theta_1^2(t)\,dt^2 + \sigma_{11}^2(t)\,d^2W_1(t) + \sigma_{22}^2(t)\,d^2W_2(t) + 2\theta_1(t)\,dt\,\sigma_{11}(t)\,dW_1(t)$$

$$+ 2\theta_1(t)\,dt\,\sigma_{22}(t)\,dW_2(t) + 2\sigma_{11}(t)\,\sigma_{22}(t)\,dW_1(t)\,dW_2(t)$$

$$= \left[\sigma_{11}^2(t) + \sigma_{22}^2(t)\right] dt.$$

Continue

In a similar way, we may derive the differential formulas

$$dY(t)\,dY(t) = \left(\sigma_{21}^2(t) + \sigma_{22}^2(t)\right) dt, \tag{4.6.4}$$

$$dX(t)\,dY(t) = \left(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right) dt. \tag{4.6.5}$$

Equation (4.6.5) says that, for every $T \geq 0$,

$$[X, Y](T) = \int_0^T \left(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right) dt. \tag{4.6.6}$$

Theorem 4.6.2. (Two-Dimentional Ito-Doeblin formula)

Theorem 4.6.2 (Two-dimensional Itô-Doeblin formula). *Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} are defined and are continuous. Let $X(t)$ and $Y(t)$ be Itô processes as discussed above. The* two-dimensional Itô-Doeblin formula in differential form *is*

$$
\begin{aligned}
&df\big(t, X(t), Y(t)\big) \\
&= f_t\big(t, X(t), Y(t)\big)\, dt + f_x\big(t, X(t), Y(t)\big)\, dX(t) + f_y\big(t, X(t), Y(t)\big)\, dY(t) \\
&\quad + \frac{1}{2} f_{xx}\big(t, X(t), Y(t)\big)\, dX(t)\, dX(t) + f_{xy}\big(t, X(t), Y(t)\big)\, dX(t)\, dY(t) \\
&\quad + \frac{1}{2} f_{yy}\big(t, X(t), Y(t)\big)\, dY(t)\, dY(t). \qquad (4.6.8)
\end{aligned}
$$

Before discussing formula (4.6.8), we rewrite it, leaving out t wherever possible, to obtain the same formula in the more compact notation

$$
\begin{aligned}
df(t, X, Y) = {} & f_t\, dt + f_x\, dX + f_y\, dY \\
& + \frac{1}{2} f_{xx}\, dX\, dX + f_{xy}\, dX\, dY + \frac{1}{2} f_{yy}\, dY\, dY. \qquad (4.6.9)
\end{aligned}
$$

proof:

$$df(t, X(t), Y(t)) = f_t(t, X(t), Y(t))\,dt + f_x(t, X(t), Y(t))\,dX_t + f_y(t, X(t), Y(t))\,dY_t$$

$$+ \frac{1}{2} f_{tt}(t, X(t), Y(t))\,dt^2 + \frac{1}{2} f_{xx}(t, X(t), Y(t))\,dX_t^2 + \frac{1}{2} f_{yy}(t, X(t), Y(t))\,dY_t^2$$

$$+ \frac{1}{2}(f_{ty} + f_{yt})(t, X(t), Y(t))\,dt\,dY_t + \frac{1}{2}(f_{tx} + f_{xt})(t, X(t), Y(t))\,dt\,dX_t + \frac{1}{2}(f_{xy} + f_{yx})(t, X(t), Y(t))\,dX_t\,dY_t$$

Since $dt^2 = 0$, $dt\,dY_t = dt\,dX_t = 0$.

$$\Rightarrow df(t, X(t), Y(t)) = f_t\,dt + f_x\,dX + f_y\,dY + \frac{1}{2} f_{xx}\,dX^2 + \frac{1}{2} f_{yy}\,dY^2 + f_{xy}\,dX\,dY$$

Note: Since f_{xy}, f_{yx} are both continuous and defined, so we have

$$f_{xy} = \lim_{k \to 0} \frac{f_y(x+k, y) - f_y(x, y)}{k} = \lim_{k \to 0} \lim_{h \to 0} \frac{(f(x+k, y+h) - f(x+k, y)) - (f(x, y+h) - f(x, y))}{kh}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{(f(x+k, y+h) - f(x, y+h)) - (f(x+k, y) - f(x, y))}{kh} = \lim_{h \to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h}$$

$$= f_{yx}.$$

Corollary 4.6.3. (Ito product rule)

$$df(t, X, Y) = f_t\,dt + f_x\,dX + f_y\,dY + \frac{1}{2} f_{xx}\,dX\,dX + f_{xy}\,dX\,dY + \frac{1}{2} f_{yy}\,dY\,dY. \qquad (4.6.9)$$

Corollary 4.6.3 (Itô product rule). *Let $X(t)$ and $Y(t)$ be Itô processes. Then*

$$d\big(X(t)Y(t)\big) = X(t)\,dY(t) + Y(t)\,dX(t) + dX(t)\,dY(t).$$

PROOF: In (4.6.9), take $f(t,x,y) = xy$, so that $f_t = 0$, $f_x = y$, $f_y = x$, $f_{xx} = 0$, $f_{xy} = 1$, and $f_{yy} = 0$. □

Theorem 4.6.4. (Levy’s One-Dimentional Characterization Theorem)

Theorem 4.6.4 (Lévy, one dimension). *Let $M(t)$, $t \geq 0$, be a martingale relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.*

proof(idea):

- Lemma
- Euler formula: $\exp(ix) = \cos(x)+i\sin(x)$, where i is imaginary
- $M(t)-M(s) \sim N(0,t-s)$
- $M(t1)-M(t2), M(t2)-M(t3), M(t3)-M(t4), \ldots$ are independent

proof:

<u>Lemma</u> Let $M(t)$ be as in Thm 4.6.4 and let $f \in C(\mathbb{R}^2)$ such that f, f', f'' are bounded

$\forall\, 0 \le s \le t$, we have $E(f(M_t)|\mathcal{F}_s) = f(M_s) + \frac{1}{2}\int_s^t E(f''(M_u)|\mathcal{F}_s)\,du$

<u>pf</u> Let $\Pi = \{s = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of $[s,t]$. By Taylor's formula:

$f(X_t) = f(X_s) + \sum_{k=1}^{n}(f(X_{t_k}) - f(X_{t_{k-1}}))$, and applying Taylor's formula on $f(X_{t_k}) - f(X_{t_{k-1}})$

$$f(M_t) = f(M_s) + \sum_{k=1}^{n} f'(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^{n} f''(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2 + R_\Pi \quad \cdots (*)$$

We note that as $\|\Pi\| \to 0$, $R_\Pi \to 0$ since $[M,M](t) = t$, so $(M_{t_k} - M_{t_{k-1}})^k$ for $k \geq 3$

$= (M_{t_k} - M_{t_{k-1}})^2 \cdot (M_{t_k} - M_{t_{k-1}})^{k-2} = (t_k - t_{k-1})(M_{t_k} - M_{t_{k-1}})^{k-2} \le \|\Pi\| \cdot (M_{t_k} - M_{t_{k-1}})^{k-2} = 0$.

Since M_t has continuous path, $M_{t_k} - M_{t_{k-1}} < \infty$ for bounded interval.

By Law of total expectation (For $F_s \subseteq F_t \subseteq \mathcal{F}$, $E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s)$)

Take conditional expectation on $\mathcal{F}_s$ to $(*)$ we have:

$$E(f(M_t)|\mathcal{F}_s) = f(M_s) + \sum_{k=1}^{n} E\Big(E(f'(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}))|\mathcal{F}_{t_{k-1}}|\mathcal{F}_s\Big) + \frac{1}{2}\sum_{k=1}^{n} E\Big(E(f''(M_{t_{k-1}}))$$
$$\cdot (M_{t_k} - M_{t_{k-1}})^2 |\mathcal{F}_{t_{k-1}}|\mathcal{F}_s\Big) = f(M_s) + \frac{1}{2}\sum_{k=1}^{n} E(f''(M_{t_{k-1}})|\mathcal{F}_s)(t_k - t_{k-1}).$$

since M_t has continuous path, $E((M_{t_k} - M_{t_{k-1}})|\mathcal{F}_{t_{k-1}}) = 0$, $E\Big(E(f''(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2|\mathcal{F}_{t_{k-1}})|\mathcal{F}_s\Big)$

$= E\Big(f''(M_{t_{k-1}})(t_k - t_{k-1})|\mathcal{F}_s\Big) = E(f''(M_{t_{k-1}})|\mathcal{F}_s)(t_k - t_{k-1})$.

As $\|\Pi\| \to 0$. $f(M_s) + \frac{1}{2}\sum_{k=1}^{n} E(f''(M_{t_{k-1}})|\mathcal{F}_s)(t_k - t_{k-1}) \to f(M_s) + \frac{1}{2}\int_s^t E(f''(M_u)|\mathcal{F}_s)\,du$.

proof(continue):

pf) We would like to show $\forall\, 0\le s\le t$, $E(\exp(iu(M_t-M_s))|\mathcal{F}_s)=\exp(-\frac{1}{2}u^2(t-s))$, $\forall u\in\mathbb{R}$.

Let $f(x)=\exp(iux)$. $\forall s\le t$, $E(\exp(iuM_t)|\mathcal{F}_s)=\exp(iuM_s)-\frac{1}{2}u^2\int_s^t E(\exp(iuM_u)|\mathcal{F}_s)du \cdots (*)$, since $f'(x)=u(\sin ux-i\cos ux)$, $f''(x)=u^2(-\cos ux-i\sin ux)=-u^2\exp(iux)$.

Also $(*)\Rightarrow E(\exp(iu(M_t-M_s)|\mathcal{F}_s))=1-\frac{1}{2}u^2\int_s^t E(\exp(iu(M_u-M_s))|\mathcal{F}_s)\,du$.

$\Rightarrow \frac{d}{dt}E(\exp(iu(M_t-M_s)|\mathcal{F}_s))=-\frac{1}{2}u^2E(\exp(iu(M_u-M_s))|\mathcal{F}_s)$

let $g(t)=E(\exp(iu(M_t-M_s)|\mathcal{F}_s))$, we have $g'(t)=-\frac{1}{2}u^2g(t)$, $g(s)=1$

Solving this ODE gives $E(\exp(iu(M_t-M_s))|\mathcal{F}_s)=g(t)=\exp(-\frac{1}{2}u^2(t-s))$.

If this can be shown, then we have $M_t-M_s\sim N(0,t-s)$, since the characteristic function of M_t-M_s $=E(\exp(iu(M_t-M_s)))=E(E(\exp(iu(M_t-M_s))|\mathcal{F}_s))=E(\exp(-\frac{1}{2}u^2(t-s)))=\exp(-\frac{1}{2}u^2(t-s))$.

Again by Law of Iterated Expectations and given the partition $0=t_0<t_1<\dots<t_n=t<\infty$.

$$E\left(\prod_{i=1}^n \exp(iu(M_{t_i}-M_{t_{i-1}}))|\mathcal{F}_0\right)=E\left(E\left(\prod_{i=1}^n \exp(iu(M_{t_i}-M_{t_{i-1}})|\mathcal{F}_{t_{i-1}})\right)|\mathcal{F}_0\right)$$

$$=(t_n-t_{n-1})\cdot E\left(E\left(\prod_{i=1}^{n-1}\exp(iu(M_{t_i}-M_{t_{i-1}})|\mathcal{F}_{t_{i-1}})\right)|\mathcal{F}_0\right)=\dots=\prod_{i=1}^n(t_i-t_{i-1})$$

$$=\prod_{i=1}^n E(\exp(iu(M_{t_i}-M_{t_{i-1}})))\Rightarrow \phi_Y=\prod_{i=1}^n\phi_{M_{t_i}-M_{t_{i-1}}},\ Y=(M_{t_1}-M_{t_0}, M_{t_2}-M_{t_1},\dots,M_{t_n}-M_{t_{n-1}})$$

$\Rightarrow$ $M_{t_1}, M_{t_2}-M_{t_1},\dots,M_{t_n}-M_{t_{n-1}}$ are all independent.