Stochastic Calculus For Finance - volume 2
Section 3.7 Reflection Principle
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Definition

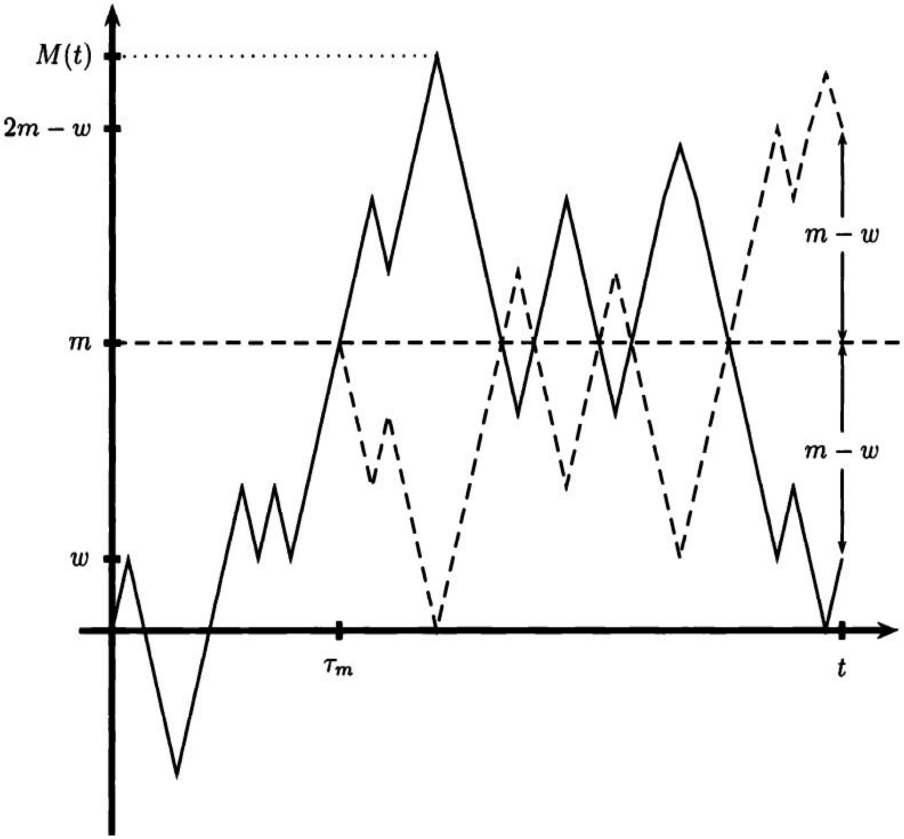
Given a random process W_n,

we define the first passage time τ_m of W_n at level m as :

$\tau_m = min\{n;\ W_n = m\}$, for a positive numer m;

If the random process W_n never reach level m,

then we define $\tau_m = \infty$.


M(t)
2m − w
m
w
τ_m
t
m − w
m − w

Goal & Application

We would like to understand the first passage time distribution, and furthermore the joint distribution of Brownian motion and its maximum.

The FPTD arises from some walks (processes) such that something remarkable happens when it reaches a certain point (absorbing point).

The distribution is used in pricing exotic options, such as barrier options.

Recall

From volume 1, we have studied the discrete case; we use counting method and reflection principle to derive the FPTD for the discrete case.

As for continuous case, since we can approximate it by infinitely many discrete cases, so we would also like to “count” the FPTD for the continuous case.

Of course, since Brownian motion contains uncountable points, we would like to “count” it in a sense of integration.

Reflection Equality

Fix some positive real level m, time t and some difference w, where $w \le m$.

WLOG, consider a Brownian motion $M(t)$ starting at 0.

$$\mathbb{P}\left\{\tau_m \le t,\ W(t) \le w\right\} = \mathbb{P}\left\{\left(\tau_m \le t\right), W(t) \ge 2m - w\right\}, \quad w \le m,\ m > 0 \quad (3.7.1)$$

After substituting $w = m$ into formula (3.7.1), we obtain:

$$\mathbb{P}\left\{\tau_m \le t,\ W(t) \le m\right\} = \mathbb{P}\{W(t) \ge m\}.$$

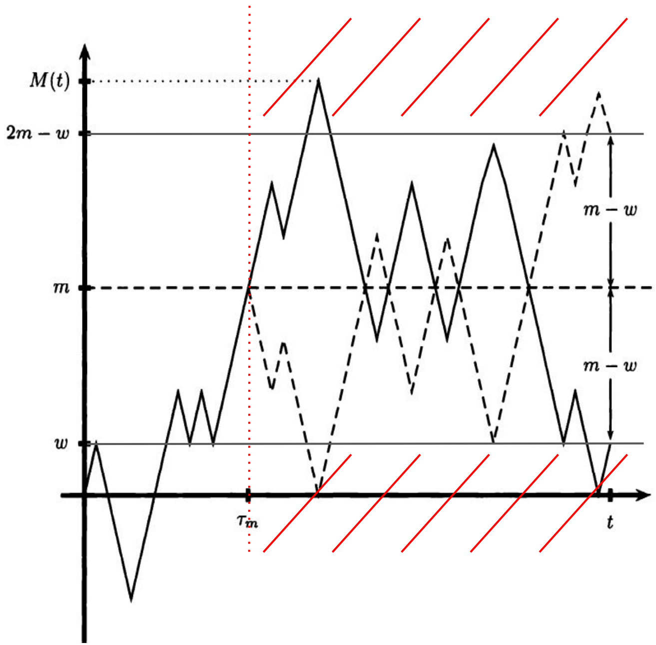
But also if $W(t) \ge m$, then $\tau_m \le t$ naturally holds, hence:

$$\mathbb{P}\left\{\tau_m \le t,\ W(t) \ge m\right\} = \mathbb{P}\{W(t) \ge m\}.$$

Together we have the cumulative distribution function for τ_m:

$$\mathbb{P}\left\{\tau_m \le t\right\} = \mathbb{P}\left\{\tau_m \le t,\ W(t) \le m\right\} + \mathbb{P}\left\{\tau_m \le t,\ W(t) \ge m\right\}$$

$$= 2\mathbb{P}\{W(t) \ge m\} \qquad (*)$$



Theorem 3.7.1.

For all $m \neq 0$, *the random variable* τ_m *has cumulative distribution function*

$$\mathbb{P}\left\{\tau_m \leq t\right\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} \mathrm{d}y, \ t \geq 0, \qquad (3.7.2)$$

and density

$$f_{\tau_m}(t) = \frac{d}{dt}\mathbb{P}\left\{\tau_m \leq t\right\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, \ t \geq 0. \qquad (3.7.3)$$

proof

From (*) , $\mathbb{P}\{\tau_m \le t\} = 2\mathbb{P}\{W(t) \ge m\} = \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}} dx,$

we then make the change of variable $y = \frac{|x|}{\sqrt{t}}$ and this gives $\frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^\infty e^{-\frac{y^2}{2}} dy.$

To obtain (3.7.3) , we differentiate (3.7.2) with respect to t on both sides:

$$\frac{d}{dt}\mathbb{P}\{\tau_m \le t\} = \frac{2}{\sqrt{2\pi}} \times \frac{d}{dt}\left(\lim_{a \to \infty} \int_{\frac{|m|}{\sqrt{t}}}^{a} e^{-\frac{y^2}{2}} dy \right)$$

$$= \frac{2}{\sqrt{2\pi}} \times \left(-e^{-\frac{m^2}{2t}} \right) \times \left(-\frac{|m|}{2t\sqrt{t}} \right) = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}$$

Remark 3.7.2.

We would like to first introduce the Laplace transform for suitable functions f :

For $t \geq 0,\ F(\alpha) := L\{f(t)\} := \int_0^\infty f(t)\, e^{-\alpha t} \mathrm{d}t,\ \forall \alpha \in \mathbb{C}.$

From (3.7.3), we can obtain the Laplace transform formula for τ_m:

$$\mathbb{E}\{e^{-\alpha\tau_m}\} = \int_0^\infty e^{-\alpha t} f_{\tau_m}(t)\, \mathrm{d}t = \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \text{ for all } \alpha > 0. \qquad (3.7.4)$$

Notes

Formula (3.6.9) provides that $\mathbb{E}\{e^{-\alpha\tau_m}\} = e^{-|m|\sqrt{2\alpha}}$ for all $\alpha > 0$.

While (3.7.4) provides that $\mathbb{E}\{e^{-\alpha\tau_m}\} = \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t$ for all $\alpha > 0$.

These two formula provides different Laplace transform formula, but with some further considerations, we can prove that these two are equivalent.

Notes

The benefit of using Laplace transform is that we can differentiate both sides wuth respect to α and we can obtain:

$$\frac{d}{d\alpha}\mathbb{E}\{e^{-\alpha\tau_m}\} = \frac{d}{d\alpha}e^{-|m|\sqrt{2\alpha}} \Rightarrow \mathbb{E}\{-\tau_m e^{-\alpha\tau_m}\} = -\frac{|m|}{\sqrt{2\alpha}}e^{-|m|\sqrt{2\alpha}}$$

$$\Rightarrow \mathbb{E}\{\tau_m e^{-\alpha\tau_m}\} = \frac{|m|}{\sqrt{2\alpha}}e^{-|m|\sqrt{2\alpha}} \Rightarrow \mathbb{E}\{\tau_m\} \rightarrow \infty\ ,\ as\ \alpha \rightarrow 0.$$

Notes

Or equivalently:

$$-\frac{d}{d\alpha}F(\alpha) = L\{t \times f(t)\} = L\left\{\frac{|m|}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}\right\},$$

$$\Rightarrow \mathbb{E}\left\{\tau_m e^{\alpha \tau_m}\right\} = L\left\{\frac{|m|}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}\right\},$$

$$\Rightarrow \lim_{\alpha \to 0} \mathbb{E}\left\{\tau_m e^{\alpha \tau_m}\right\} = \lim_{\alpha \to 0} L\left\{\frac{|m|}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}\right\} = \lim_{\alpha \to 0} \frac{e^{-\sqrt{2\alpha}}}{\sqrt{2\alpha}},$$

$$\Rightarrow \mathbb{E}\{\tau_m\} \to \infty\ ,\ as\ \alpha \to 0.$$

Distribution of Brownian Motion and Its Maximum

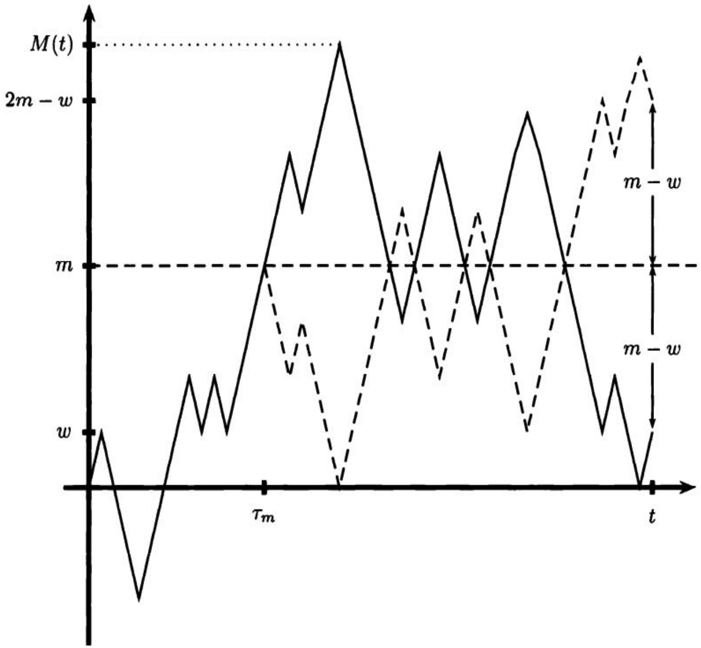
After understanding FTPD, we can now derive the joint distribution of Brownian motion and its maximum, which is used in pricing barrier options.

When pricing exotic options, it is more convenient to first simulate the value before some time $t<T$, and then stimulate the maximum of the Brownian motion due to the property of path dependence, so we would also like to derive the conditional distribution of the maximum given some information.

Definition

We difine the *maximum to date* for Brownian motion to be :

$$M(t) = \max_{0 \leq s \leq t} W(s) \qquad (3.7.5)$$



Theorem 3.7.3.

For $t > 0$, *the joint density of* $(M(t), W(t))$ *is*

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \leq m, m > 0. \qquad (3.7.7)$$

proof

We first observe that M(t) $\geq m$ if and only if $\tau_m \leq t$. So we can rewrite (3.7.1) as

$$\mathbb{P}\{M(t) \geq m, W(t) \leq w\} = \mathbb{P}\{W(t) \geq 2m - w\},\ w \leq m,\ m > 0. \quad (3.7.6)$$

Since $\mathbb{P}\{\mathrm{M(t)} \geq \mathrm{m}, \mathrm{W(t)} \leq \mathrm{w}\} = \int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x, y)\, dydx$

and $\mathbb{P}\{W(t) \geq 2m - w\} = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}} dz,$

hence we get $\int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x, y)\, dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}} dz.$

proof

From $\int_{m}^{\infty} \int_{-\infty}^{w} f_{M(t),W(t)}(x,y)\,dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^{\infty} e^{-\frac{z^2}{2t}}\,\mathrm{d}z,$ we can then

differentiate first with respect to m to obtain:

$$\frac{d}{dm} \lim_{a\to\infty} \left(\int_{m}^{a} \int_{-\infty}^{w} f_{M(t),W(t)}(x,y)\,dydx \right) = \frac{1}{\sqrt{2\pi t}} \frac{d}{dm} \lim_{b\to\infty} \left(\int_{2m-w}^{b} e^{-\frac{z^2}{2t}}\,\mathrm{d}z \right),$$

and we get $-\int_{-\infty}^{w} f_{M(t),W(t)}(m,y)\,dy = -\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}},$

proof

Finally we differentiate both sides with respect to w and obtain:

$$\frac{d}{dw}\lim_{a\to\infty}\left(-\int_{-a}^{w} f_{M(t),W(t)}(m,y)\,dy\right)=\frac{d}{dw}-\frac{2}{\sqrt{2\pi t}}e^{-\frac{(2m-w)^2}{2t}},$$

and so we get:

$$-f_{M(t),W(t)}(m,w)=-\frac{2(2m-w)}{t\sqrt{2\pi t}}e^{-\frac{(2m-w)^2}{2t}},$$

or $f_{M(t),W(t)}(m,w)=\dfrac{2(2m-w)}{t\sqrt{2\pi t}}e^{-\frac{(2m-w)^2}{2t}}.\qquad(3.7.7)$

Corollary 3.7.4.

The conditional distribution of $M(t)$ *given* $W(t)=w$ *is*

$$f_{M(t)|W(t)}(m|w)=\frac{2(2m-w)}{t}e^{-\frac{2m(m-w)}{t}},w\leq m,\ m>0.$$

proof

The conditional density can be written as:

$$f_{M(t)|W(t)}(m|w) = \frac{f_{M(t),W(t)}(m,w)}{f_{W(t)}(w)}$$

$$= \frac{2(2m-w)}{t\sqrt{2\pi t}} \times e^{-\frac{(2m-w)^2}{2t}} \div \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}}$$

$$= \frac{2(2m-w)}{t\sqrt{2\pi t}} \times \sqrt{2\pi t}\, e^{-\frac{(2m-w)^2}{2t}+\frac{w^2}{2t}}$$

$$= \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}.$$