Stochastic Calculus For Finance - volume 2

Section 8.4 Finite-Expiration American Put

8.4.1 Analytical Characterization of the Put Price

8.4.2 Probabilistic Characterization of the Put Price

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Definition 8.4.1. Let $0 \leq t \leq T$ and $x \geq 0$ be given. Assume $S(t) = x$.
Let $\mathrm{F}_u^{(t)}$, $t \leq u \leq T$, denote the σ-algebra generated by the process $S(v)$ as v ranges over $[t, u]$ and let $\mathcal{T}_{t,T}$ denote the set of stopping times for the filtration $\mathrm{F}_u^{(t)}, t \leq u \leq T$, taking values in $[t, T]$ or taking the value ∞.
In other words, $\{\tau \leq u\} \in \mathrm{F}_u^{(t)}$ for every $u \in [t, T]$;a stopping time in $\mathcal{T}_{t,T}$ makes the decision to stop at a time $u \in [t, T]$ based only on the path of the stock price between times t and u.

The price at time t of the American put expiring at time T is defined to be

$$v(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} \tilde{E}\left[e^{-r(\tau - t)}\big(K - S(\tau)\big)\middle| S(t) = x\right] \qquad (8.4.1)$$

In the event that $\tau = \infty$, we interpret $e^{-r(\tau - t)}(K - S(\tau))$ to be zero. This is the case when the put expires unexercised.

8.4.1 Analytical Characterization of the Put Price

The finite-expiration American put price function $v(t,x)$ satisfies the linear complementarity conditions (cf. (8.3.18)–(8.3.20))

$v(t,x) \geq (K-x)^+$ for all $t \in [0,T], x \geq 0$ (8.4.2)

$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$ for all $t \in [0,T), x \geq 0$ (8.4.3)

and for each $t \in [0,T)$ and $x \geq 0$, equality holds in either (8.4.2) or (8.4.3). (8.4.4)

As with the perpetual American put, the owner of the finite-expiration American put should wait until the stock price falls to a certain level at or below K before exercising, but now this level $L(T-t)$ depends on the time to expiration $T-t$.

The level L_* for the perpetual American put is $\lim_{T\to\infty} L(T)$

At the other extreme, $L(0) = K$; at expiration,

stock price < K: exercise ; stock price > K: not exercise ; stock price =K: indifferent

No formula is known for the function $L(T-t)$, but this function can be determined numerically from the analytic characterization of the put price provided below.

$L(T-t)$ decreases with increasing $(T-t)$, as shown in Figure 8.4.1.

The set $\{(t,x); 0 \le t \le T, x \ge 0\}$ can be divided into two regions

stopping set:
$\mathrm{S} = \{(t,x); v(t,x) = (K-x)^+\}$ (8.4.5)

continuation set:
$\mathrm{C} = \{(t,x); v(t,x) > (K-x)^+\}$ (8.4.6)

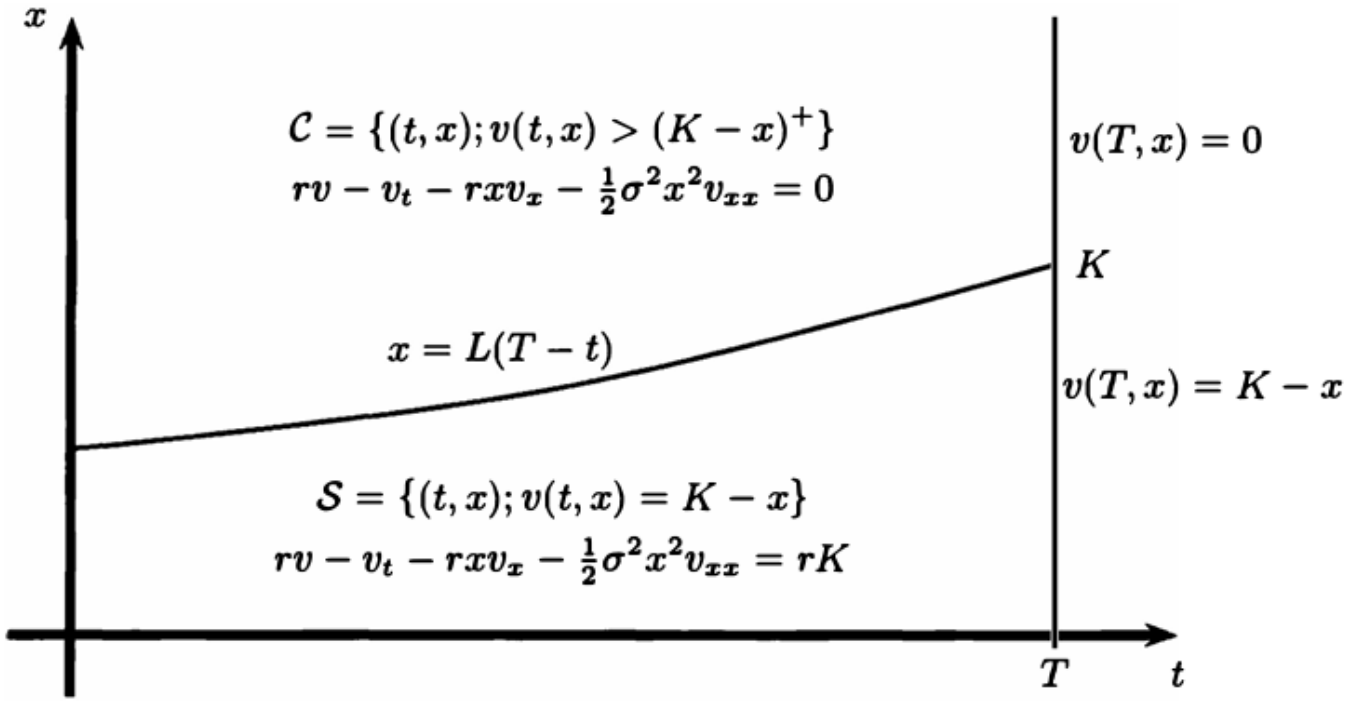


Fig. 8.4.1. Finite-expiration American put.

The function $x = L(T-t)$ forms the boundary between C and S and belongs to S.

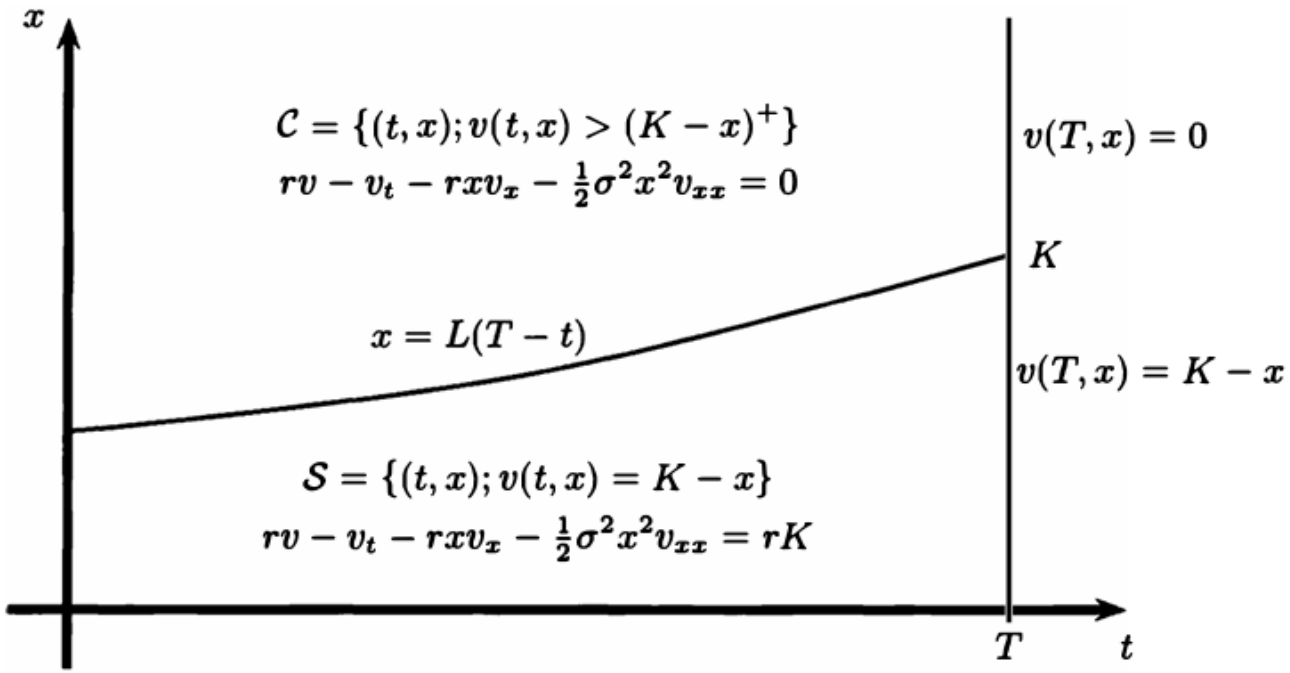
Because of (8.4.4), equality holds in (8.4.3) for (t,x) in C, $t \neq T$.
For (t,x) in S, strict inequality holds in (8.4.3) except on the curve $x = L(T-t)$, where equality holds in (8.4.3).
Because $v(t,x) = (K-x)^+ = K - x$ for $0 \le x \le L(T-t)$ we have (see Figure 8.4.1)

$$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rK \;\; for\; x \in S$$

$v(t,x) \ge (K-x)^+$ for all $t \in [0,T], x \ge 0$ (8.4.2)

$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \ge 0$
for all $t \in [0,T), x \ge 0$ (8.4.3)

equality holds in either (8.4.2) or (8.4.3). (8.4.4)



Because $v(t,x) = K - x$ for $0 \le x \le L(T-t)$, we also have the left-hand derivative $v_x(t, x-) = -1$ on the curve $x = L(T-t)$.

The put price $v(t,x)$ satisfies the smooth-pasting condition that $v_x(t,x)$ is continuous, even at $x = L(T-t)$.

In other words $v_x(t, x+) = v_x(t, x-) = -1$ for $x = L(T-t)$, $0 \le t < T$ (8.4.7)

The smooth-pasting condition does not hold at $t=T$. Indeed,

$$L(0) = K \text{ and } v(t,x) = (K-x)^+ \quad (8.4.8), v(t,x) = (K-x)^+ = \begin{cases} K - x, x < K \\ 0, \qquad x \ge K \end{cases}$$

so $v_x(t, x-) = -1$, whereas $v_x(t, x+) = 0$ for $x = L(0)$.

Also, $v_t(t,x)$ and $v_{xx}(t,x)$ are not continuous along the curve $x = L(T-t)$.

The equations $rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0, x \ge L(T-t)$, $v(t,x) = K - x,\ 0 \le x \le L(T-t)$ together with the smooth-pasting condition (8.4.7), the terminal condition (8.4.8), and the asymptotic condition $\lim_{x\to\infty} v(t,x)=0$ (8.4.9) determine the function $v(t,x)$. Using these equations, one can set up a finite-difference scheme to simultaneously compute $v(t,x)$ and $L(T-t)$.

8.4.2 Probabilistic Characterization of the Put Price

Theorem 8.4.2. Let $S(u), t \leq u \leq T$, be the stock price of (8.3.1) starting at $S(t) = x$ and with the stopping set S defined by (8.4.5). Let

$$\tau_* = \min\{u \in [t,T]; (u, S(u)) \in S\} \qquad (8.4.10)$$

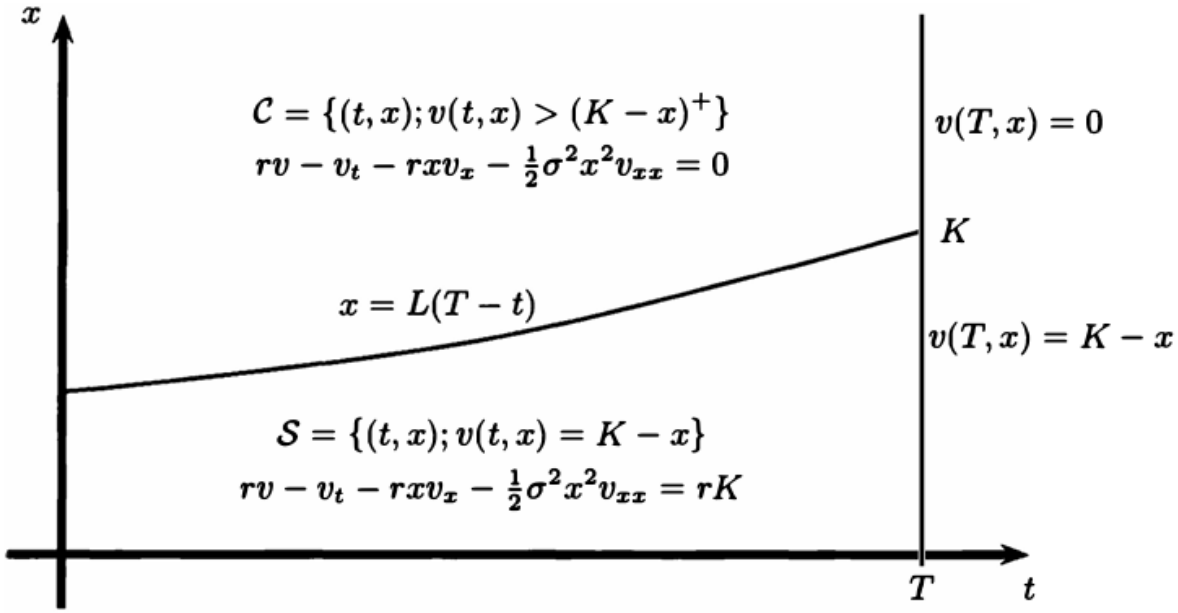
where we interpret τ_* to be ∞ if $(u, S(u))$ doesn't enter S for any $u \in [t, T]$.
Then $e^{-ru} v\big(u, S(u)\big), t \leq u \leq T$, is a supermartingale under $\tilde{P}$, and the stopped process $e^{-r(u \wedge \tau_*)} v\big(u, S(u \wedge \tau_*)\big), t \leq u \leq T$ is a martingale.

Proof.The Itô–Doeblin formula applies to $e^{-ru} v\big(u, S(u)\big)$, even though $v_u(u, x)$ and $v_{xx}(u, x)$ are not continuous along the curve $x = L(T - u)$ because the process $S(u)$ spends zero time on this curve. All that is needed for the Itô–Doeblin formula to apply is that $v(u, x)$ be continuous , and this follows from the smooth-pasting condition (8.4.7). We may thus compute

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

$$d\left[e^{-ru}v\big(u,S(u)\big)\right]$$
$$= e^{-ru}\left[-rv\big(u,S(u)\big)du + v_u\big(u,S(u)\big)du + v_x\big(u,S(u)\big)dS(u) + \frac{1}{2}v_{xx}\big(u,S(u)\big)dS(u)dS(u)\right]$$
$$= e^{-ru}\left[-rv\big(u,S(u)\big) + v_u\big(u,S(u)\big) + rS(u)v_x\big(u,S(u)\big) + \frac{1}{2}\sigma^2S^2(u)v_{xx}\big(u,S(u)\big)\right]du$$
$$+ e^{-ru}\sigma S(u)v_x\big(u,S(u)\big)d\widetilde{W}(u) \qquad (8.4.11)$$
$$= -e^{-ru}rK\mathbb{I}_{\{S(u)<L(T-u)\}}du + e^{-ru}\sigma S(u)v_x\big(u,S(u)\big)d\widetilde{W}(u)$$

According to Figure 8.4.1,
the du term in (8.4.11) is $-e^{-ru}rK\mathbb{I}_{\{S(u)<L(T-u)\}}$.
This is nonpositive, and so $e^{-ru}v\big(u,S(u)\big)$
is a supermartingale under $\widetilde{\mathbb{P}}$.



In fact, starting from $u=t$ and up until the time τ_*,
we have $S(u)>L(T-u)$, so the du term is zero.
Therefore, the stopped process
$e^{-r(u\wedge\tau_*)}v\big(u,S(u\wedge\tau_*)\big), t \le u \le T$, is a martingale.

Corollary 8.4.3.

Consider an agent with initial capital $X(0) = v(0, S(0))$, the initial finite-expiration put price. Suppose this agent uses the portfolio process $\Delta(u) = v_x(u, S(u))$ and consumes cash at rate $C(u) = rK\mathbb{I}_{\{S(u)<L(T-u)\}}$ per unit time. Then $X(u) = v(u, S(u))$ for all times u between u=0 and the time the option is exercised or expires. In particular, $S(u) \geq \big(K - S(u)\big)^+$ for all times u until the option is exercised or expires, so the agent can pay off a short option position regardless of when the option is exercised.

Proof. The differential of the agent's discounted portfolio value is given by (8.3.24). Substituting for $\Delta(u)$ and $C(u)$ in this equation and comparing it to (8.4.11), we see that

$$d[e^{-ru}X(u)] = d\big[e^{-ru}v\big(u, S(u)\big)\big]$$

$$= -e^{-ru}rK\mathbb{I}_{\{S(u)<L(T-u)\}}du + e^{-ru}\sigma S(u)v_x\big(u, S(u)\big)d\widetilde{W}(u).$$

Integrating this equation and using $X(0) = v(0, S(0))$, we obtain $X(t) = v(t, S(t))$ for all times t prior to exercise or expiration.

PROOF: The differential of the agent's portfolio value process is

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt - C(t)\,dt,$$

so the differential of the discounted portfolio value process is

$$\begin{aligned} d\big(e^{-rt}X(t)\big) &= e^{-rt}\big(-rX(t)\,dt + dX(t)\big) \\ &= e^{-rt}\big(\Delta(t)\,dS(t) - r\Delta(t)S(t)\,dt - C(t)\,dt\big) \\ &= e^{-rt}\big(\Delta(t)\sigma S(t)\,d\widetilde{W}(t) - C(t)\,dt\big). \end{aligned} \tag{8.3.24}$$

Remark 8.4.4.

The proofs of Theorem 8.4.2 and Corollary 8.4.3 use the analytic characterization of the American put price captured in Figure 8.4.1 plus the smooth-pasting condition that guarantees that $v_x(t,x)$ is continuous even on the curve $x = L(T-t)$ so that the Itô–Doeblin formula can be applied.

Here we show that the only function $v(t,x)$, satisfying these conditions, is the function $v(t,x)$ defined by (8.4.1). To do this, we first fix $0 \le t \le T$.The supermartingale property for $e^{-rt}v(t,S(t))$ of Theorem 8.4.2 and Theorem 8.2.4 (optional sampling) implies that

$$e^{-r(t\wedge\tau)}v\big(t,S(t\wedge\tau)\big) \ge \tilde{E}[e^{-r(T\wedge\tau)}v(T\wedge\tau,S(T\wedge\tau))|F(t)]$$

For $\tau \in \mathcal{T}_{t,T}$we have $t\wedge\tau = t$, whereas $\mathrm{T}\wedge\tau = \tau$ if $\tau < \infty$ and $\mathrm{T}\wedge\tau = \mathrm{T}$ if $\tau = \infty$.

Therefore, for $\tau \in \mathcal{T}_{t,T}$,

$$\begin{aligned} e^{-rt}v\big(t,S(t)\big) &\ge \tilde{E}[e^{-r\tau}v\big(\tau,S(\tau)\big)\mathbb{I}_{\{\tau<\infty\}} + e^{-r(T)}v\big(T,S(T)\big)\mathbb{I}_{\{\tau=\infty\}}|F(t)] \\ &\ge \tilde{E}[e^{-r\tau}v\big(\tau,S(\tau)\big)|F(t)] \qquad (8.4.12) \end{aligned}$$

where, as usual, we interpret $e^{-r\tau}v\big(\tau,S(\tau)\big) = 0$ if $\tau = \infty$.

Inequality (8.4.2) and the fact that $v\big(t,S(t)\big) \ge \big(K - S(t)\big)^+ \ge K - S(t)$ imply that

$$\tilde{E}\big[e^{-r\tau}v\big(\tau,S(\tau)\big)\big|F(t)\big] \ge \tilde{E}[e^{-r\tau}(K-S(\tau))|F(t)] \qquad (8.4.13)$$

Putting (8.4.12) and (8.4.13) together, we conclude that

$$e^{-rt}v\big(t,S(t)\big) \geq \tilde{E}[e^{-r\tau}(K-S(\tau))|F(t)] \qquad (8.4.14)$$

Because S(t) is a Markov process, the right-hand side of (8.4.14) is a function of t and S(t). In particular, if we denote the value of S(t) by x, we may rewrite (8.4.14) as

$$e^{-rt}v(t,x) = \tilde{E}[e^{-r\tau}\big(K-S(\tau)\big)|S(t)=x] \qquad (8.4.15)$$

Since (8.4.15) holds for any $\tau \in \mathcal{T}_{t,T}$, we conclude that

$$v(t,x) \geq \max_{\tau\in\mathcal{T}_{t,T}} \tilde{E}[e^{-r(\tau-t)}\big(K-S(\tau)\big)|S(t)=x] \qquad (8.4.16)$$

For the reverse inequality, we recall from Theorem 8.4.2 that the stopped process $e^{-r(u\wedge\tau_*)}v\big(u,S(u\wedge\tau_*)\big)$ is a martingale, where τ_* defined by (8.4.10) is such that $v\big(\tau_*,S(\tau_*)\big) = \big(K-S(\tau_*)\big)$ if $\tau_* < \infty$. Replacing τ by τ_* in (8.4.12), we make the first inequality into an equality. If $\tau_* = \infty$, we have $(T,S(T)) \in C$ (i.e., $S(T) > K$), so $v\big(T,S(T)\big)\mathbb{I}_{\{\tau_*=\infty\}} = 0$. This makes the second inequality in (8.4.12) into an equality. Finally, because $v(\tau_*,S(\tau_*)) = K - S(\tau_*)$ on $\mathbb{I}_{\{\tau_*<\infty\}}$ the inequality in (8.4.13) is an equality, and hence (8.4.15) becomes

$$v(t,x) = \tilde{E}[e^{-r(\tau_*-t)}\big(K-S(\tau_*)\big)|S(t)=x] \qquad (8.4.17)$$

Equation (8.4.17) shows that equality must hold in (8.4.16), and this is (8.4.1)