Stochastic Calculus For Finance - volume 2

Section 7.4.4 Computation of the Price of the Look back Option

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$S(t) = S(0)e^{\sigma\widehat{W}(t)}, Y(t) = \max_{0\le u\le t} S(u) = S(0)e^{\sigma\widehat{M}(t)}, \widehat{M}(t) = \max_{0\le u\le t} \widehat{W}(u)$

Payoff $= Y(T) - S(T)$

$Y(T) = S(0)e^{\sigma\widehat{M}(\mathrm{T})+\sigma\widehat{M}(t)-\sigma\widehat{M}(t)} = S(0)e^{\sigma\widehat{M}(t)}e^{\sigma\widehat{M}(T)-\sigma\widehat{M}(t)} = Y(t)e^{\sigma(\widehat{M}(T)-\widehat{M}(t))}$

$$\widehat{M}(T) - \widehat{M}(t) = \begin{cases} \max_{t\le s\le T} \widehat{W}(s) - \widehat{M}(t), if \max_{t\le s\le T} \widehat{W}(s) > \widehat{M}(t) \\ 0, if \max_{t\le s\le T} \widehat{W}(s) < \widehat{M}(t) \end{cases} = \left[\max_{t\le s\le T} \widehat{W}(s) - \widehat{M}(t)\right]^+$$

$$= \left[\max_{t\le s\le T} \widehat{W}(s) - \widehat{W}(t) - \widehat{M}(t) + \widehat{W}(t)\right]^+ = \left[\max_{t\le s\le T}(\widehat{W}(s) - \widehat{W}(t)) - (\widehat{M}(t) - \widehat{W}(t))\right]^+$$

$$= \left[\max_{t\le s\le T}(\widehat{W}(s) - \widehat{W}(t)) - (\frac{1}{\sigma}\log\frac{Y(t)}{S(0)} - \frac{1}{\sigma}\log\frac{S(t)}{S(0)})\right]^+$$

$$= \left[\max_{t\le s\le T}(\widehat{W}(s) - \widehat{W}(t)) - \frac{1}{\sigma}\log\frac{Y(t)}{S(t)}\right]^+$$

Risk-neutral pricing formula

$$V(t)e^{-rt} = \tilde{E}[V(T)e^{-rT}|F(t)] = e^{-rT}\tilde{E}[Y(T) - S(T)|F(t)]$$

$$V(t) = e^{-r(T-t)}\tilde{E}[Y(T) - S(T)|F(t)]$$
$$= e^{-r(T-t)}\tilde{E}[Y(T)|F(t)] - e^{-r(T-t)}\tilde{E}[S(T)|F(t)]$$
$$= e^{-r(T-t)}\tilde{E}\left[Y(t)e^{\left[\max_{t\le s\le T}\sigma(\widehat{W}(s)-\widehat{W}(t))-\log\frac{Y(t)}{S(t)}\right]^+}\middle|F(t)\right] - S(t)$$
$$= e^{-r(T-t)}Y(t)\tilde{E}\left[e^{\left[\max_{t\le s\le T}\sigma(\widehat{W}(s)-\widehat{W}(t))-\log\frac{Y(t)}{S(t)}\right]^+})\middle|F(t)\right] - S(t)$$

Because Y(t) and S(t) are F(t)-measurable and $\sigma \max_{t\le s\le T}(\widehat{W}(s) - \widehat{W}(t))$ is independent of F(t), we can use the Independence Lemma to write the conditional expectation as g(S(t), Y(t)), where

$$g(x,y) = \tilde{E}\{\exp\left(\left[\max_{t\le s\le T}\sigma(\widehat{W}(s) - \widehat{W}(t)) - \log\frac{y}{x}\right]^+\right)\}$$

$$V(t) = e^{-r(T-t)}Y(t)g(S(t), Y(t)) - S(t),$$
$$\mathrm{v}(t, x, y) = e^{-r(T-t)}yg(x, y) - x$$

$\max_{t \le s \le T}(\widehat{W}(s) - \widehat{W}(t))$ has the same unconditional distribution under $\tilde{P}$ as $\max_{0 \le u \le \tau}(\widehat{W}(u) - \widehat{W}(0)) = \widehat{M}(\tau), \tau = T - t$,the function g(x, y) can also be written as

$$g(x, y) = \tilde{E}\{\exp\left(\left[\max_{t \le s \le T}\sigma(\widehat{W}(s) - \widehat{W}(t)) - \log\frac{y}{x}\right]^+\right)\}$$
$$= \tilde{E}\{\exp\left(\left[\sigma\widehat{M}(\tau) - \log\frac{y}{x}\right]^+\right)\} = \begin{cases} 1, if \widehat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x} \\ e^{\sigma\widehat{M}(\tau) + \log\frac{x}{y}} = \frac{x}{y}e^{\sigma\widehat{M}(\tau)}, if \widehat{M}(\tau) \ge \frac{1}{\sigma}\log\frac{y}{x} \end{cases}$$

$$g(x,y) = \tilde{E}\left[1 \times \mathbb{I}_{\widehat{M}(\tau)\le\frac{1}{\sigma}\log\frac{y}{x}} + \frac{x}{y}e^{\sigma\widehat{M}(\tau)} \times \mathbb{I}_{\widehat{M}(\tau)\ge\frac{1}{\sigma}\log\frac{y}{x}}\right]$$

$$= \underline{\tilde{P}\left(\widehat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\right)} + \tilde{E}\left[\frac{x}{y}e^{\sigma\widehat{M}(\tau)} \times \mathbb{I}_{\widehat{M}(\tau)\ge\frac{1}{\sigma}\log\frac{y}{x}}\right]$$

Recall $\tilde{P}\big(\widehat{M}(\tau) \le m\big) = N\left(\frac{m-\alpha T}{\sqrt{T}}\right) - e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right)$

$S(t) = S(0)e^{\sigma[\widetilde{W}(t)+\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)]}$, $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$, $\delta_\pm(\tau,s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$

$$\frac{m-\alpha T}{\sqrt{T}} = \frac{\frac{1}{\sigma}\log\frac{y}{x} - \frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sqrt{\tau}} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{y}{x} - \left(r-\frac{1}{2}\sigma^2\right)\tau\right] = \frac{-1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{y} + \left(r-\frac{1}{2}\sigma^2\right)\tau\right] = -\delta_-\left(\tau,\frac{x}{y}\right)$$

$$\frac{-m-\alpha T}{\sqrt{T}} = \frac{-\frac{1}{\sigma}\log\frac{y}{x} - \frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sqrt{\tau}} = \frac{1}{\sigma\sqrt{\tau}}\left[-\log\frac{y}{x} - \left(r-\frac{1}{2}\sigma^2\right)\tau\right] = \frac{-1}{\sigma\sqrt{\tau}}\left[\log\frac{y}{x} + \left(r-\frac{1}{2}\sigma^2\right)\tau\right] = -\delta_-\left(\tau,\frac{y}{x}\right)$$

$$e^{2\alpha m} = e^{2\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\times\frac{1}{\sigma}\log\frac{y}{x}} = \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}$$

➔ $\underline{\tilde{P}\left(\widehat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\right) = N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) - \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)}$

$$g(x,y) = \tilde{P}\left(\widehat{M}(\tau) \leq \frac{1}{\sigma}\log\frac{y}{x}\right) + \tilde{E}\left[\frac{x}{y}e^{\sigma\widehat{M}(\tau)} \times \mathbb{I}_{\widehat{M}(\tau)\geq\frac{1}{\sigma}\log\frac{y}{x}}\right]$$

Recall density function

$$\tilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right)$$

$$\tilde{E}\left[\frac{x}{y}e^{\sigma\widehat{M}(\tau)} \times \mathbb{I}_{\widehat{M}(\tau)\geq\frac{1}{\sigma}\log\frac{y}{x}}\right] = \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} e^{\sigma m}\tilde{f}_{\widehat{M}(\tau)}(m)dm$$

$$= \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}dm - \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm$$

$$r\tau - \frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2 = r\tau - \frac{1}{2\tau}(m-\alpha\tau)^2 + \sigma(m-\alpha\tau) - \frac{1}{2}\sigma^2\tau$$

$$= r\tau - \frac{1}{2\tau}(m-\alpha\tau)^2 + \sigma\left(m - \frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\tau\right) - \frac{1}{2}\sigma^2\tau$$

$$= r\tau - \frac{1}{2\tau}(m-\alpha\tau)^2 + \sigma m - r\tau + \frac{1}{2}\sigma^2\tau - \frac{1}{2}\sigma^2\tau = \sigma m - \frac{1}{2\tau}(m-\alpha\tau)^2$$

$$\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}dm$$

$$=\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\frac{2}{\sqrt{2\pi\tau}}e^{r\tau-\frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2}dm=\frac{x}{y}\frac{2e^{r\tau}}{\sqrt{2\pi\tau}}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}e^{-\frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2}dm$$

Change of variable $\xi=\frac{\alpha\tau+\sigma\tau-m}{\sqrt{\tau}}$

$$\underline{\frac{x}{y}\frac{2e^{r\tau}}{\sqrt{2\pi\tau}}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}e^{-\frac{1}{2\tau}(m-\alpha\tau-\sigma\tau)^2}dm=\frac{x}{y}\frac{2e^{r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\delta_+\left(\tau,\frac{x}{y}\right)}e^{-\frac{1}{2}(\xi)^2}d\xi=\frac{2xe^{r\tau}}{y}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)}$$

$$d\xi=\frac{-dm}{\sqrt{\tau}}\Rightarrow dm=-\sqrt{\tau}d\xi$$

$$\frac{\alpha\tau+\sigma\tau-\frac{1}{\sigma}\log\frac{y}{x}}{\sqrt{\tau}}=\frac{1}{\sqrt{\tau}}(\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\tau-\frac{1}{\sigma}\log\frac{y}{x})=\frac{1}{\sigma}\frac{1}{\sqrt{\tau}}(\log\frac{x}{y}+r\tau+\frac{1}{2}\sigma^2\tau)=\delta_+\left(\tau,\frac{x}{y}\right)$$

$$\tilde{E}\left[\frac{x}{y}e^{\sigma\widehat{M}(\tau)}\times\mathbb{I}_{\widehat{M}(\tau)\geq\frac{1}{\sigma}\log\frac{y}{x}}\right]=\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\frac{2}{\sqrt{2\pi\tau}}e^{\sigma m-\frac{1}{2\tau}(m-\alpha\tau)^2}dm-\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm$$

$$-\frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}2\alpha e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm$$

$$=-\frac{2\alpha x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}e^{(\sigma+2\alpha)m}N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right)dm$$

$$=-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty}\int_{-\infty}^{\frac{-m-\alpha\tau}{\sqrt{\tau}}}e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}d\xi\,dm$$

$$=-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau}e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}dm\,d\xi$$

$$(\sigma+2\alpha)m=\left(\sigma+2\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\tau\right)m=\frac{2}{\sigma}rm$$

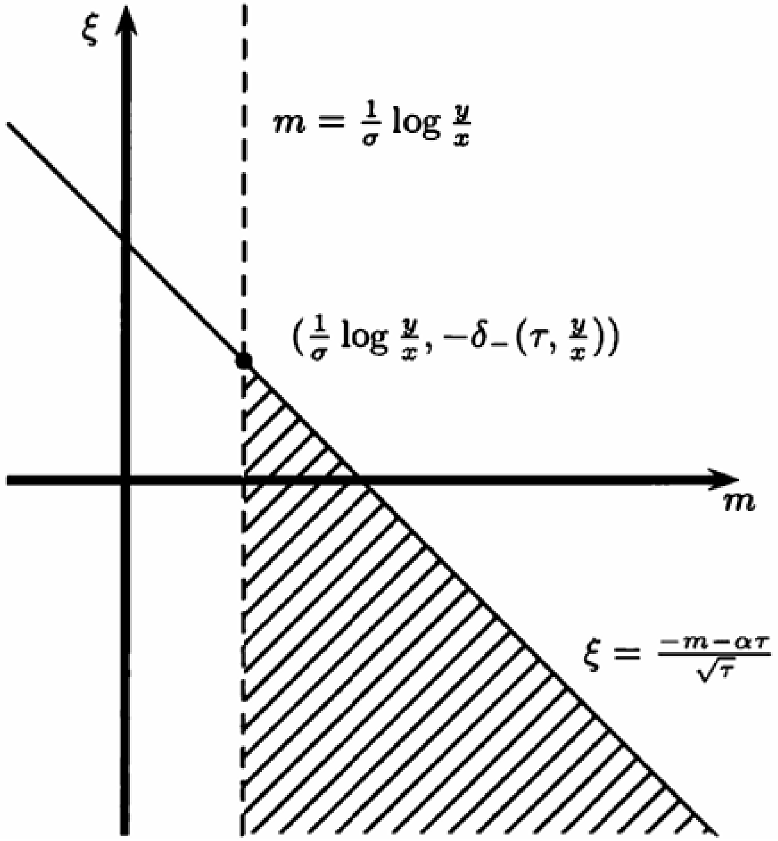

Fig. 7.4.2. Reversal of integration in (7.4.32).

$$-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau} e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}\,dm\,d\xi$$

$$\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau} e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}\,dm=\frac{\sigma}{2r}e^{\frac{2rm}{\sigma}-\frac{1}{2}\xi^2}\Bigg|_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau}$$

$$=\frac{\sigma}{2r}e^{\boxed{\frac{2r(-\xi\sqrt{\tau}-\alpha\tau)}{\sigma}-\frac{1}{2}\xi^2}}-\frac{\sigma}{2r}e^{\frac{2r\frac{1}{\sigma}\log\frac{y}{x}}{\sigma}-\frac{1}{2}\xi^2}=\frac{\sigma}{2r}e^{r\tau-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2}-\frac{\sigma}{2r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}}e^{-\frac{1}{2}\xi^2}$$

$$\frac{2r(-\xi\sqrt{\tau}-\alpha\tau)}{\sigma}-\frac{1}{2}\xi^2=-\frac{1}{2}\xi^2-\frac{2r\xi\sqrt{\tau}}{\sigma}-\frac{2r\alpha\tau}{\sigma}=-\frac{1}{2}\xi^2-\frac{2r\xi\sqrt{\tau}}{\sigma}-\frac{2r\alpha\tau}{\sigma}$$

$$=-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+\frac{2r^2\tau}{\sigma^2}-\frac{2r\alpha\tau}{\sigma}=-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+\frac{2r\tau}{\sigma^2}(r-\sigma\alpha)$$

$$=-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+\frac{2r\tau}{\sigma^2}\left(r-\sigma\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\right)=-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2+r\tau$$

$$-\frac{2\alpha x}{y\sqrt{2\pi}}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{-\xi\sqrt{\tau}-\alpha\tau} e^{\frac{2}{\sigma}rm-\frac{1}{2}\xi^2}\,dm\,d\xi$$

$$=-\frac{\alpha\sigma x}{ry\sqrt{2\pi}}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)} e^{r\tau-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2}d\xi+\frac{\sigma\alpha x}{r\sqrt{2\pi}}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)} e^{-\frac{1}{2}\xi^2}d\xi$$

$$=-\frac{\alpha\sigma x e^{r\tau}}{ry\sqrt{2\pi}}\int_{-\infty}^{-\delta_-\left(\tau,\frac{y}{x}\right)} e^{-\frac{1}{2}\left(\xi+\frac{2r\sqrt{\tau}}{\sigma}\right)^2}d\xi+\frac{\sigma\alpha}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)$$

$$=-\frac{\alpha\sigma x}{ry}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)+\frac{\sigma\alpha}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)$$

Change of variable $\eta=\xi+\frac{2r\sqrt{\tau}}{\sigma}$

$$-\delta_-\left(\tau,\frac{y}{x}\right)+\frac{2r\sqrt{\tau}}{\sigma}=\frac{1}{\sigma\sqrt{\tau}}\left[-\log\frac{y}{x}-\left(r-\frac{1}{2}\sigma^2\right)\tau+2r\tau\right]=\delta_+\left(\tau,\frac{x}{y}\right)$$

$$\mathrm{v}(t,x,y) = e^{-r(T-t)} y g(x,y) - x$$

$$g(x,y)$$

$$= \tilde{P}\left(\widehat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\right) + \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{\sigma m - \frac{1}{2\tau}(m-\alpha\tau)^2} dm$$

$$- \frac{x}{y}\int_{\frac{1}{\sigma}\log\frac{y}{x}}^{\infty} 2\alpha e^{(\sigma+2\alpha)m} N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right) dm$$

$$= N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) - \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) + \frac{2xe^{r\tau}}{y} N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)$$

$$- \frac{\alpha\sigma x}{ry} e^{r\tau} N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) + \frac{\sigma\alpha}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)$$

代入$\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$

$$
\begin{aligned}
&g(x,y)\\
&= N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) - \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) + 2\frac{x}{y}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)\\
&- \frac{\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\sigma}{r}\frac{x}{y}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) + \frac{\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)\sigma}{r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)\\
&= N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) - \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) + 2\frac{x}{y}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right)\\
&- \left(1-\frac{\sigma^2}{2r}\right)\frac{x}{y}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) + \left(1-\frac{\sigma^2}{2r}\right)\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)\\
&= N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) + \left(1+\frac{\sigma^2}{2r}\right)\frac{x}{y}e^{r\tau}N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) - \frac{\sigma^2}{2r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)
\end{aligned}
$$

$$\mathrm{v}(t,x,y) = e^{-r(T-t)} y g(x,y) - x$$

$$= e^{-r\tau} y \left[N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) + \left(1+\frac{\sigma^2}{2r}\right)\frac{x}{y} e^{r\tau} N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) - \frac{\sigma^2}{2r}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right)\right] - x$$

$$= e^{-r\tau} y N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) + \left(1+\frac{\sigma^2}{2r}\right) x N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) - \frac{\sigma^2}{2r} e^{-r\tau}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} x N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) - x$$

$, 0 \le t \le T, 0 < x \le y$； Recall $\mathrm{v}(t,x,y) = \mathrm{y}\mathrm{u}\left(t,\frac{x}{y}\right)$

➔ $\mathrm{u}\left(t,\frac{x}{y}\right) =$

$$e^{-r\tau} N\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) + \left(1+\frac{\sigma^2}{2r}\right)\frac{x}{y} N\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) - \frac{\sigma^2}{2r} e^{-r\tau}\left(\frac{x}{y}\right)^{-\frac{2r}{\sigma^2}}\frac{x}{y} N\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) - \frac{x}{y}$$

$$z = \frac{x}{y},$$

$$\mathrm{u}(t,z) = e^{-r\tau} N\big(-\delta_-(\tau,z)\big) + \left(1+\frac{\sigma^2}{2r}\right) z N\big(\delta_+(\tau,z)\big) - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N\big(-\delta_-(\tau,z)\big) - z$$