Stochastic Calculus For Finance - volume 2

Section 5.6.2 Futures Contracts
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Section 5.6.2 Futures Contracts

Consider a time interval $[0, T]$,which we divide into subintervals using the partition points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. We shall refer to each subinterval $[t_k, t_{k+1})$ as a “day.”
Suppose the interest rate is constant within each day. Then the discount process is given by $D(0) = 1$ and, for k=0,1,…,n−1,

$$D(t_{k+1}) = \exp\left\{ -\int_0^{t_{k+1}} R(u)\,du \right\} = \exp\left\{ -\sum_{j=0}^{k} R(t_j)(t_{j+1} - t_j) \right\},$$

which is $F(t_k)$-measurable.

According to the risk-neutral pricing formula (5.6.1), the zero-coupon bond paying 1 at maturity T has time-t_k price

$$B(t_k, T) = \frac{1}{D(t_k)} \tilde{E}[D(T) \mid F(t_k)].$$

An asset whose price at time t is $S(t)$ has time- t_k forward price

$$\mathrm{For}_s(t_k, T) = \frac{S(t_k)}{B(t_k, T)}$$

Suppose we take a long position in the forward contract at time t_k (i.e., agree to receive S(T) and pay $For_s(t_k, T)$ at time T). The value of this position at time $t_j \geq t_k$ is

$$V\left(t_j, T\right)D\left(t_j\right) = \tilde{E}\left[V(T, T)D(T)\middle|F\left(t_j\right)\right]$$

$$V(t_j, T) = \frac{1}{D\left(t_j\right)}\tilde{E}[D(T)(S(T) - \frac{S(t_k)}{B(t_k, T)}) \mid F(t_j)]$$

$$= \frac{1}{D\left(t_j\right)}\tilde{E}[D(T)S(T) \mid F(t_j)] - \frac{S(t_k)}{B(t_k, T)} \cdot \frac{1}{D\left(t_j\right)}\tilde{E}[D(T) \mid F(t_j)]$$

$$= S(t_j) - S(t_k) \cdot \frac{B\left(t_j, T\right)}{B(t_k, T)}$$

Suppose we take a long position in the forward contract at time t_k (i.e., agree to receive S(T) and pay $For_s(t_k, T)$ at time T). The value of this position at time $t_j \geq t_k$ is

$$V(t_j, T) = S(t_j) - S(t_k) \cdot \frac{B(t_j,T)}{B(t_k,T)}, B(t,T) = e^{-r(\mathrm{T}-t)}$$

$$V\left(t_j, T\right) = S\left(t_j\right) - S(t_k) \cdot \frac{e^{-r\left(T-t_j\right)}}{e^{-r(T-t_k)}} = S\left(t_j\right) - S(t_k) \cdot e^{r\left(t_j - t_k\right)}$$

If the growth rate of S is greater than r

➔positive value for buyer, negative value for seller

➔one of the parties to the forward contract could become concerned about default by the other party (default risk)

➔futures contracts can reduce default risk by margin system

If an agent holds a long futures position between times t_k and t_{k+1} , then at time t_{k+1} he receives a payment $Fut_s(t_{k+1}, T) - Fut_s(t_k, T)$.
The sum of payments received by an agent who purchases a futures contract at time zero and holds it until delivery date T is

$$\left(Fut_s(t_1, T) - Fut_s(t_0, T)\right) + \left(Fut_s(t_2, T) - Fut_s(t_1, T)\right) + \cdots\cdots + \left(Fut_s(t_n, T) - Fut_s(t_{n-1}, T)\right)$$

$$= Fut_s(T, T) - Fut_s(t_0, T), Fut_s(T, T) = S(T)$$

$$= S(T) - Fut_s(t_0, T).$$

zero value on entry:

$$\frac{1}{D(t_k)} \tilde{E}\left[D(t_{k+1})\left(Fut_s(t_{k+1}, T) - Fut_s(t_k, T)\right) \mid F(t_k) \right]$$

$$= \frac{D(t_{k+1})}{D(t_k)} [\tilde{E}[Fut_s(t_{k+1}, T)|F(t_k)] - Fut_s(t_k, T)] = 0$$

➔ $Fut_s(t_k, T) = \tilde{E}[Fut_s(t_{k+1}, T)|F(t_k)]$ ➔martingale
$(Fut_s(t_k, T) = \tilde{E}[S(T)|F(t_k)])$

Proof: The value at time t_k of future cash flow equals to 0, $j \geq k+1$
(if Fut_s is martingale)

$$\frac{1}{D(t_k)}\tilde{E}\left[D(t_{j+1})\left(Fut_s(t_{j+1},T) - Fut_s(t_j,T)\right) \middle| F(t_k)\right]$$

$$= \frac{1}{D(t_k)}\tilde{E}\left[\tilde{E}[D(t_{j+1})\left(Fut_s(t_{j+1},T) - Fut_s(t_j,T)\right)|F(t_j)] \middle| F(t_k)\right]$$

$$= \frac{1}{D(t_k)}\tilde{E}\left[D(t_{j+1})[\tilde{E}[Fut_s(t_{j+1},T)|F(t_j)] - Fut_s(t_j,T)] \middle| F(t_k)\right] = 0$$

➔the present value of cash flow in any future time period will be 0

Theorem 5.6.5

The futures price is a martingale under the risk-neutral measure $\tilde{P}$; it satisfies $Fut_s(T,T) = S(T)$ and the value of a long (or a short) futures position to be held over an interval of time is always zero.

If the filtration, F(t),0≤t≤T, is generated by a Brownian motion W(t),0≤t≤T, then Corollary 5.3.2 of the Martingale Representation Theorem implies that

$$Fut_s(t,T) = Fut_s(0,T) + \int_0^t \tilde{\Gamma}(u)\, d\widetilde{W}(u),\ 0 \le t \le T$$

for some adapted integrand process $\tilde{\Gamma}$ (i.e., $dFut_s(t,T) = \tilde{\Gamma}(t)d\widetilde{W}(t)$).

Let $0 \le t_0 < t_1 \le T$,holds $\Delta(t)$ futures contracts at times t between t_0 and t_1. Investing and/or borrowing at the interest rate $R(t)$ prevailing at the time of the investing or borrowing.

Profit $X(t)$ from these trading satisfies

$$dX(t) = \Delta(t)dFut_s(t,T) + R(t)X(t)dt = \Delta(t)\tilde{\Gamma}(t)d\widetilde{W}(t) + R(t)X(t)dt$$

and thus $d(D(t)X(t)) = D(t)\Delta(t)\tilde{\Gamma}(t)d\widetilde{W}(t)$.

$$\begin{aligned} d\big(D(t)X(t)\big) &= \mathrm{d}D(t)X(t) + D(t)dX(t) + \mathrm{d}D(t)dX(t) \\ &= -R(t)D(t)dt \times X(t) + D(t)dX(t) + \mathrm{d}D(t)dX(t) \end{aligned}$$

Assume that at time t_0 the profit is $X(t_0) = 0$.At time t_1 the agent's profit $X(t_1)$ will satisfy $D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\tilde{\Gamma}(u)d\widetilde{W}(u)$.(5.6.7)

Because Itô integrals are martingales, we have

$\tilde{E}[\,D(t_1)X(t_1) \mid F(t_0)\,]$

$= \tilde{E}\left[\int_0^{t_1} D(u)\Delta(u)\tilde{\Gamma}(u)d\widetilde{W}(u) - \int_0^{t_0} D(u)\Delta(u)\tilde{\Gamma}(u)d\widetilde{W}(u) \,\middle|\, F(t_0)\right]$

$= \tilde{E}[\int_0^{t_1} D(u)\Delta(u)\tilde{\Gamma}(u)d\widetilde{W}(u) \mid F(t_0)] - \int_0^{t_0} D(u)\Delta(u)\tilde{\Gamma}(u)d\widetilde{W}(u) = 0$.(5.6.8)

According to the risk-neutral pricing formula, the value at time t_0 of a payment of $X(t_1)$ at time t_1 is $\frac{1}{D(t_0)}\tilde{E}[D(t_1)X(t_1) \mid F(t_0)]$,and we have just shown that this is zero.

The value of owning a long position over the interval t_0 to t_1 is obtained by setting $\Delta(u)$=1 for all u

The value of holding a short position is obtained by setting $\Delta(u)$=−1for all u.

In both cases, this value is zero.

Section 5.6.3 Forward-Futures Spread

$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}$,**r constant**➔$B(t,T) = e^{-r(\mathrm{T}-t)}$, $\mathrm{For}_S(t,T) = e^{r(\mathrm{T}-t)}S(t)$

$Fut_S(t,T) = \tilde{E}[S(T)|F(t)] = \frac{D(t)D(T)}{D(t)D(T)}\tilde{E}[S(T)|F(t)]$

$= \frac{D(t)}{D(T)} \times \frac{1}{D(t)}\tilde{E}[S(T)D(T)|F(t)] = \frac{e^{-rt}}{e^{-rT}}S(t) = e^{r(\mathrm{T}-t)}S(t)$

➔ $\mathrm{For}_S(t,T) = Fut_S(t,T)$

r is not constant➔$B(t,T) = \frac{1}{D(t_k)}\tilde{E}[\, D(T) \mid F(t_k)\,], B(0,T) = \frac{1}{1}\tilde{E}[D(T)]$

$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}$, $\mathrm{For}_S(0,T) = \frac{S(0)}{B(0,T)} = \frac{S(0)}{\tilde{E}[D(T)]}$

$Fut_S(t,T) = \tilde{E}[S(T)|F(t)], Fut_S(0,T) = \tilde{E}[S(T)]$

$S(0) \times 1 = S(0)D(0) = \tilde{E}[S(T)D(T)]$

$\mathrm{For}_S - Fut_S = \frac{S(0)}{\tilde{E}[D(T)]} - \tilde{E}[S(T)] = \frac{1}{\tilde{E}[D(T)]}[S(0) - \tilde{E}[S(T)]\tilde{E}[D(T)]]$

$= \frac{1}{\tilde{E}[D(T)]}\left[\tilde{E}[S(T)D(T)] - \tilde{E}[S(T)]\tilde{E}[D(T)]\right] = \frac{\widetilde{cov}(S(T),D(T))}{B(0,T)}$

Section 6.2 Stochastic Differential Equations

A stochastic differential equation is an equation of the form:

$$\boldsymbol{dX(u) = \beta(u, X(u))\, du + \gamma(u, X(u))\, dW(u)} \qquad (6.2.1)$$

Here $\beta(u, x)$ and $\gamma(u, x)$ are given functions, called the drift and diffusion, respectively. In addition to this equation, an initial condition of the form:

$\boldsymbol{X(t) = x}$, where $t \geq 0$ and $x \in \mathbb{R}$, is specified.

The problem is then to find a stochastic process X(T), defined for T ≥ t, such that:

$$\boldsymbol{X(T) = X(t) + \int_t^T \beta(u, X(u))\, du + \int_t^T \gamma(u, X(u))\, dW(u)} \qquad (6.2.3)$$

The solution X(T) at time T will be F(T)-measurable

Example 6.2.1 (Geometric Brownian motion).
The stochastic differential equation for geometric Brownian motion is

$$\boldsymbol{dS(u) = \alpha S(u)\,du + \sigma S(u)\,dW(u)}$$

In the notation of (6.2.1), $\beta(u,x) = \alpha x$ and $\gamma(u,x) = \sigma x$.

We know the formula for the solution to this stochastic differential equation when the initial time is zero and the initial position is $S(0)$, namely

$S(t) = S(0)e^{\left(\alpha-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}$, for $T \geq t$, $S(T) = S(0)e^{\left(\alpha-\frac{1}{2}\sigma^2\right)T+\sigma W(T)}$

Dividing $S(T)$ by $S(t)$, $\boldsymbol{\frac{S(T)}{S(t)} = exp\{\sigma(W(T)-W(t)) + (\alpha - \frac{1}{2}\sigma^2)(T-t)\}}$

If the initial condition is given at time t and $S(t)=x$, then last equation becomes

$\boldsymbol{S(T) = x \times exp\{\sigma(W(T)-W(t)) + (\alpha - \frac{1}{2}\sigma^2)(T-t)\}}$.

➔when we use the initial condition $S(t)=x$

$S(T)$ depends only on the path of the Brownian motion between times t and T.

Example 6.2.2 (Hull–White interest rate model).
Consider the stochastic differential equation

$$dR(u) = \big(a(u) - b(u)R(u)\big)du + \sigma(u)\, d\widetilde{W}(u)$$

where $a(u)$,$b(u)$and $\sigma(u)$ are nonrandom positive functions of the time variable u and $\widetilde{W}(u)$ is a Brownian motion under a risk-neutral measure $\tilde{P}$.
In this case, we use the dummy variable r rather than x, and

$$\beta(u,r) = a(u) - b(u)r, \gamma(u,r) = \sigma(u).$$

Let us take the initial condition $R(t)$=r.
We can solve the stochastic differential equation by first using the stochastic differential equation to compute

$$d\left(e^{\int_0^u b(v)dv} R(u)\right) = e^{\int_0^u b(v)dv}\big(b(u)R(u)du + dR(u)\big)$$

$$= e^{\int_0^u b(v)dv}(a(u)du + \sigma(u)\, d\widetilde{W}(u))$$

Integrating both sides from t to T and using the initial condition R(t)=r, we obtain the formula

$$e^{\int_0^T b(v)dv}\mathrm{R(T)}$$
$$= re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv}\, a(u)du + \int_t^T e^{\int_0^u b(v)dv}\, \sigma(u)\, d\widetilde{W}(u)$$

,which we can solve for R(T):

$$\mathrm{R(T)} = re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv}\, a(u)du + \int_t^T e^{-\int_u^T b(v)dv}\, \sigma(u)\, d\widetilde{W}(u).$$

This is an explicit formula for the solution R(T). The right-hand side of the final equation does not involve the interest rate process R(u) apart from the initial condition R(t)=r; it contains only this initial condition, an integral with respect to time, and an Itô integral of given functions. Note also that the Brownian motion between times t and T only enters this formula.

Recall from Theorem 4.4.9 that the Itô integral $\int_t^T e^{-\int_u^T b(v)dv} \sigma(u)\, d\widetilde{W}(u)$ of the nonrandom integrand $e^{-\int_u^T b(v)dv}\sigma(u)$ is normally distributed with mean zero and variance $\int_t^T e^{-2\int_u^T b(v)dv} \sigma^2(u)\, du$.

The other terms appearing in the formula above for R(T) are nonrandom. Therefore, under the risk-neutral measure $\tilde{P}$, R(T) is normally distributed with

mean: $re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du$

variance: $\int_t^T e^{-2\int_u^T b(v)dv} \sigma^2(u)\, du$

In particular, there is a positive probability that R(T) is negative. This is one of the principal objections to the Hull–White model.

Theorem 4.4.9 (Itô integral of a deterministic integrand). *Let $W(s)$, $s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a* nonrandom *function of time. Define $I(t) = \int_0^t \Delta(s)\, dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)\, ds$.*