Stochastic Calculus For Finance - volume 2

Section 5.2.1 Girsanov’s Theorem for a Single Brownian Motion
Section 5.2.2 Stock Under the Risk-Neutral Measure

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Section 5.2.1 Girsanov's Theorem for a Single Brownian Motion

In Theorem 1.6.1, we began with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a nonnegative r.v Z satisfying $\mathbb{E}_{\mathbb{P}} Z = 1$. We then defined a new probability measure $\widetilde{\mathbb{P}}$ by the formula

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) dP(\omega) \quad \text{for all } A \in \mathcal{F}. \tag{5.2.1}$$

Any random variable X now has two expectations, one under the original probability measure $\mathbb{P}$, which we denote $\mathbb{E}X$, and the other under the new probability measure $\widetilde{\mathbb{P}}$, which we denote $\widetilde{\mathbb{E}}X$. These are related by the formula $\widetilde{\mathbb{E}}X = \mathbb{E}[XZ]$ (5.2.2).

If $\mathbb{P}\{Z > 0\} = 1$, then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent measure i.e.$\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$ and (5.2.2) has the companion formula $\mathbb{E}X = \widetilde{\mathbb{E}}[\frac{X}{Z}]$ (5.2.3)

We say Z is the *Radon-Nikodym derivative* of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write $Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$

Suppose further that Z is an almost surely positive random variable satisfying $\mathbb{E}Z = 1$, and we define $\widetilde{\mathbb{P}}$ by (5.2.1). We can then define the *Radon-Nikodym derivative process*

$$Z(t) = \mathbb{E}[Z|\mathcal{F}(t)], 0 \leq t \leq T$$

Where $\{\mathcal{F}(t): t \geq 0\}$ is some given filtration.

The *Radon-Nikodym derivative process* is a martingale because of iterated conditioning

(Theorem 2.3.2(iii)): for $0 \leq s \leq t \leq T$,

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}\big[\mathbb{E}[Z|\mathcal{F}(t)]|\mathcal{F}(s)\big] = \mathbb{E}[Z|\mathcal{F}(s)] = Z(s). \tag{5.2.7}$$

Lemma 5.2.1. Let t satisfying $0 \leq t \leq T$ be given and let Y be an F(t)-measurable random variable. Then

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)] \qquad (5.2.8)$$

Proof:

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ] = \mathbb{E}[\,\mathbb{E}[YZ|F(t)]\,] = \mathbb{E}\big[Y\mathbb{E}[Z|F(t)]\big] = \mathbb{E}[YZ(t)]$$

Lemma 5.2.2. Let s and t satisfying $0 \leq s \leq t \leq T$ be given and let Y be an F(t)-measurable random variable. Then

$$\widetilde{\mathbb{E}}[Y|F(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|F(s)] \quad (5.2.9)$$

Vertify the two defining conditions of conditional expectation (Definition 2.3.1)

(i) $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|F(s)]$ is F(s)-measurable

(ii) Now let $A \in F(s)$, we want to show

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|F(s)]d\widetilde{\mathbb{P}} = \int_A Y\, d\widetilde{\mathbb{P}} = \widetilde{\mathbb{E}}[\mathbb{I}_A Y]$$

Definition 2.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies*

(i) (Measurability) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable, and

(ii) (Partial averaging)

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega)d\mathbb{P}(\omega) = \int_A X(\omega)d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{G}. \qquad (2.3.17)$$

(ii) Now let $A \in F(s)$, we want to show

$$\int_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|F(s)] d\widetilde{\mathbb{P}} = \int_A Y \, d\widetilde{\mathbb{P}} = \widetilde{\mathbb{E}}[\mathbb{I}_A Y]$$

$$\widetilde{\mathbb{E}}\left[\mathbb{I}_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|F(s)] \right]$$

$= \widetilde{\mathbb{E}}\left[\mathbb{E}\left[\mathbb{I}_A \frac{1}{Z(s)} YZ(t)|F(s)\right] \right]$,use lemma 5.2.1 ($\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]$)

$$= \mathbb{E}\left[Z(s)\mathbb{E}\left[\mathbb{I}_A \frac{1}{Z(s)} YZ(t)|F(s)\right]\right] = \mathbb{E}\big[\mathbb{E}[\mathbb{I}_A YZ(t)|F(s)]\big] = \mathbb{E}[\mathbb{I}_A YZ(t)]$$

$$= \widetilde{\mathbb{E}}[\mathbb{I}_A Y] = \int_A Y \, d\widetilde{\mathbb{P}}$$

So, $\frac{1}{Z(s)} \mathbb{E}[YZ(t)|F(s)] = \widetilde{\mathbb{E}}[Y|F(s)]$

Theorem 5.2.3 (Girsanov, one dimension). *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}, \tag{5.2.11}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du, \tag{5.2.12}$$

and assume that[1]

$$\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty. \tag{5.2.13}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by (5.2.1), the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

(i)$\mathbb{E}$Z=1

Proof:

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\}, \tag{5.2.11}$$

$$X(t) = -\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du$$

and $f(x) = e^x$ so that $f'(x) = e^x$ and $f''(x) = e^x$, we have

$$\begin{aligned} dZ(t) &= df\big(X(t)\big) \\ &= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t) \\ &= e^{X(t)}\Big(-\Theta(t)\,dW(t) - \frac{1}{2}\Theta^2(t)\,dt\Big) + \frac{1}{2}e^{X(t)}\Theta^2(t)\,dt \\ &= -\Theta(t)Z(t)\,dW(t). \end{aligned}$$

Integrating both sides of the equation above, we see that

$$Z(t) = Z(0) - \int_0^t \Theta(u)Z(u)\,dW(u). \tag{5.2.14}$$

Because Itô integrals are martingales, $Z(t)$ is a martingale. In particular, $\mathbb{E}Z = \mathbb{E}Z(T) = Z(0) = 1$.

(ii) The process $\{\widetilde{W}(t): 0 \leq t \leq T\}$ is a Brownian motion under $\widetilde{\mathbb{P}}$, where $d\widetilde{\mathbb{P}} = Zd\mathbb{P} = Z(T)d\mathbb{P}$

Proof:

Theorem 4.6.4 (Levy, one dimension). *Let $M(t)$, $t \geq 0$, be a martingale relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.*

(i) when $t = 0$, $\widetilde{W}(0) = 0$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du,$$

(ii) $\widetilde{W}(t)$ has continuous paths

(iii) $d\widetilde{W}(t)d\widetilde{W}(t) = (dW(t) + \Theta(t)dt)^2 = dW(t)dW(t) = dt$

➔only need to show that $\{\widetilde{W}(t): 0 \leq t \leq T\}$ is martingale under $\widetilde{\mathbb{P}}$

$\{\widetilde{W}(t): 0 \le t \le T\}$ is martingale under $\widetilde{\mathbb{P}}$

Proof:

Showing that $\{\widetilde{W}(t)Z(t): 0 \le t \le T\}$ is martingale under $\mathbb{P}$, then using Lemma 5.2.2 to prove that $\{\widetilde{W}(t): 0 \le t \le T\}$ is martingale under $\widetilde{\mathbb{P}}$

using Itô's product rule (Corollary 4.6.3):

$$\begin{aligned} d\big(\widetilde{W}(t)Z(t)\big) &= \widetilde{W}(t)\,dZ(t) + Z(t)\,d\widetilde{W}(t) + d\widetilde{W}(t)\,dZ(t) \\ &= -\widetilde{W}(t)\Theta(t)Z(t)\,dW(t) + Z(t)\,dW(t) + Z(t)\,\Theta(t)\,dt \\ &\quad + \big(dW(t) + \Theta(t)\,dt\big)\big(-\Theta(t)Z(t)\,dW(t)\big) \\ &= \big(-\widetilde{W}(t)\Theta(t) + 1\big)Z(t)\,dW(t). \end{aligned}$$

$$\begin{aligned} dZ(t) &= df\big(X(t)\big) \\ &= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t) \\ &= e^{X(t)}\Big(-\Theta(t)\,dW(t) - \frac{1}{2}\Theta^2(t)\,dt\Big) + \frac{1}{2}e^{X(t)}\Theta^2(t)\,dt \\ &= -\Theta(t)Z(t)\,dW(t). \end{aligned}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du,$$

$$d\widetilde{W}(t) = dW(t) + \Theta(t)dt$$

➔ $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$

$\{\widetilde{W}(t): 0 \le t \le T\}$ is martingale under $\widetilde{\mathbb{P}}$

Proof:

Showing that $\{\widetilde{W}(t)Z(t): 0 \le t \le T\}$ is martingale under $\mathbb{P}$, then using Lemma 5.2.2 to prove that $\{\widetilde{W}(t): 0 \le t \le T\}$ is martingale under $\widetilde{\mathbb{P}}$

Now let $0 \le s \le t \le T$ be given. Lemma 5.2.2 and the martingale property for $\widetilde{W}(t)Z(t)$ under $\mathbb{P}$ imply

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\widetilde{W}(s)Z(s) = \widetilde{W}(s).$$

Lemma 5.2.2. *Let s and t satisfying $0 \le s \le t \le T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]. \tag{5.2.9}$$

➔ $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$

Section 5.2.2 Stock Under the Risk-Neutral Measure

Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Here T is a fixed final time. Consider a stock price process whose differential is

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \quad 0 \le t \le T. \qquad (5.2.15)$$

The mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are allowed to be adapted processes. We assume that, for all $t \in [0,T]$, $\sigma(t)$ is almost surely not zero.

This stock price is a generalized geometric Brownian motion (see Example 4.4.8, in particular, (4.4.27)), and an equivalent way of writing (5.2.15) is (see (4.4.26))

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}. \qquad (5.2.16)$$

Recall 4.4.8

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds.$$

$$dX(t) = \sigma(t)\,dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt,$$

$$\begin{aligned} dS(t) &= df(X(t)) \\ &= f'(X(t))\,dX(t) + \frac{1}{2}f''(X(t))\,dX(t)\,dX(t) \\ &= S(0)e^{X(t)}\,dX(t) + \frac{1}{2}S(0)e^{X(t)}\,dX(t)\,dX(t) \\ &= S(t)\,dX(t) + \frac{1}{2}S(t)\,dX(t)\,dX(t) \\ &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t). \end{aligned}$$

Interest rate process $\{R(t): 0 \leq t \leq T\}$ ($R(t)$ is F(t)-measurable)

Define the discount process $\{\mathrm{D}(t): 0 \leq t \leq T\}$

$$\mathrm{D}(t) = e^{-\int_0^t R(s)ds} \qquad (5.2.17)$$

Let $I(t) = \int_0^t R(s)ds$ ➔ $dI(t) = R(t)dt,\ dI(t)dI(t) = 0$

$f(x) = e^{-x}, f'(x) = -e^{-x}, f''(x) = e^{-x}$

$$\begin{aligned} dD(t) &= df\big(I(t)\big) \\ &= f'\big(I(t)\big)\,dI(t) + \frac{1}{2}f''\big(I(t)\big)\,dI(t)\,dI(t) \\ &= -f\big(I(t)\big)R(t)\,dt \\ &= -R(t)D(t)\,dt. \end{aligned}$$

Discounted price process $\{\mathrm{D}(t)S(t): 0 \leq t \leq T\}$

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \underline{R(s)} - \frac{1}{2}\sigma^2(s)\right)ds\right\}, \qquad (5.2.19)$$

Discounted price process $\{D(t)S(t): 0 \le t \le T\}$

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\}, \tag{5.2.19}$$

using Itô's product rule (Corollary 4.6.3):

$$d\big(D(t)S(t)\big) = D(t)dS(t) + S(t)dD(t) + dS(t)dD(t) \quad (\to 0)$$
$$= D(t)\alpha(t)S(t)dt + D(t)\sigma(t)S(t)dW(t) + S(t)(-R(t)D(t)dt)$$
$$= [\alpha(t) - R(t)]D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t)$$

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)$$
$$dD(t) = -R(t)D(t)\,dt.$$

Let $\theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$, rewriting the above, we got:

$$d\big(D(t)S(t)\big) = \sigma(t)\theta(t)D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t)$$
$$= \sigma(t)D(t)S(t)[\theta(t)dt + dW(t)] = \sigma(t)D(t)S(t)d\widetilde{W}(t)$$

Theorem 5.2.3 (Girsanov, one dimension).

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du, \quad d\widetilde{W}(t) = dW(t) + \Theta(t)dt$$

Now change our measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$

➔ $\{D(t)S(t): 0 \le t \le T\}$ becomes a martingale under $\widetilde{\mathbb{P}}$

$$d\big(D(t)S(t)\big) = \sigma(t)D(t)S(t)d\widetilde{W}(t)$$

$$D(t)S(t) = D(0)S(0) + \int_0^t \sigma(t)D(t)S(t)d\widetilde{W}(t)$$

by Girsanov's theorem, $\widetilde{W}(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}$

price process $S(t)$ in terms of $\widetilde{W}(t)$

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t);\ d\widetilde{W}(t) = dW(t) + \Theta(t)dt$$

$$= \alpha(t)S(t)dt + \sigma(t)S(t)[d\widetilde{W}(t) - \Theta(t)dt]$$

$$= [\alpha(t) - \sigma(t)\Theta(t)]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

$$= R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,d\widetilde{W}(s) + \int_0^t \left(R(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\}.$$

$$\theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$