Stochastic Calculus For Finance - volume 2

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02/25/2025,卓伯呈

Section 4.4.2 Formula for Ito Process

Theorem 4.4.6 (Itô-Doeblin formula for an Itô process). *Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,*

$$\begin{aligned} f\big(T, X(T)\big) &= f\big(0, X(0)\big) + \int_0^T f_t\big(t, X(t)\big)\,dt + \int_0^T f_x\big(t, X(t)\big)\,dX(t) \\ &\quad + \frac{1}{2}\int_0^T f_{xx}\big(t, X(t)\big)\,d[X,X](t) \\ &= f\big(0, X(0)\big) + \int_0^T f_t\big(t, X(t)\big)\,dt + \int_0^T f_x\big(t, X(t)\big)\Delta(t)\,dW(t) \\ &\quad + \int_0^T f_x\big(t, X(t)\big)\Theta(t)\,dt + \frac{1}{2}\int_0^T f_{xx}\big(t, X(t)\big)\Delta^2(t)\,dt. \end{aligned} \tag{4.4.22}$$

$$df\big(t, X(t)\big) = f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t). \tag{4.4.23}$$

Ito Processes-Definition

Definition 4.4.3. Let $W(t), t \geq 0,$ be a Brownian motion, and let $F(t), t \geq 0,$ be an associated filtration. An Ito process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \theta(u)du \quad (4.4.16)$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\theta(u)$ are adapted stochastic processes

$$dX(t) = \Delta(t)dW(t) + \theta(t)dt$$

Ito Processes- quadratic variation

Lemma 4.4.4 The quadratic variation of the Ito process is

$$[X,X](t) = \int_0^t \Delta^2(u)du \quad (4.4.17)$$

Proof:

Notation $I(t) = \int_0^t \Delta(u)dW(u), R(t) = \int_0^t \theta(u)du$

$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \theta(u)du \quad (4.4.16)$

➔ $X(t) - X(0) = I(t) + R(t)$

$$\sum_{j=0}^{n-1}[X(t_{j+1}) - X(t_j)]^2$$

$$= \underbrace{\sum_{j=0}^{n-1}[I(t_{j+1}) - I(t_j)]^2}_{1} + \underbrace{\sum_{j=0}^{n-1}[R(t_{j+1}) - R(t_j)]^2}_{2} + \underbrace{2\sum_{j=0}^{n-1}[I(t_{j+1}) - I(t_j)][R(t_{j+1}) - R(t_j)]}_{3}$$

Ito Processes- quadratic variation

$$\sum_{j=0}^{n-1}\left[I(t_{j+1})-I(t_j)\right]^2$$

Recall Theorem 4.3.1(vi)

Theorem 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \le t \le T$, be an adapted stochastic process that satisfies (4.3.1). Then $I(t) = \int_0^t \Delta(u)\,dW(u)$ defined by (4.3.3) has the following properties.*

(i) **(Continuity)** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*

(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)\,dW(u)$ and $J(t) = \int_0^t \Gamma(u)\,dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))\,dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)\,dW(u)$.*

(iv) **(Martingale)** *$I(t)$ is a martingale.*

(v) **(Itô isometry)** *$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\,du$.*

(vi) **(Quadratic variation)** *$[I, I](t) = \int_0^t \Delta^2(u)\,du$.*

Ito Processes- quadratic variation

$$\sum_{j=0}^{n-1}\left[R(t_{j+1}) - R(t_j)\right]^2$$

$$\max_{0\le k\le n-1}\left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1}\left|R(t_{j+1}) - R(t_j)\right|$$

$$= \max_{0\le k\le n-1}\left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1}\left|\int_{t_j}^{t_{j+1}} \Theta(u)\,du\right|$$

$$\le \max_{0\le k\le n-1}\left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |\Theta(u)|\,du$$

$$0$$

$$= \max_{0\le k\le n-1}\left|R(t_{k+1}) - R(t_k)\right| \cdot \int_0^t |\Theta(u)|\,du,$$

$as\|\pi\| \to 0 \ (R \ is \ continuous)$

Ito Processes- quadratic variation

$$2\sum_{j=0}^{n-1}[I(t_{j+1})-I(t_j)][R(t_{j+1})-R(t_j)]$$

$$2\max_{0\le k\le n-1}\left|I(t_{k+1})-I(t_k)\right|\cdot\sum_{j=0}^{n-1}\left|R(t_{j+1})-R(t_j)\right|$$

$$\le 2\max_{0\le k\le n-1}\left|I(t_{k+1})-I(t_k)\right|\cdot\int_0^t|\Theta(u)|\,du,$$

$\to 0$ $as\ \|\pi\|\to 0\ (I\ is\ continuous)$

$\Rightarrow [X,X](t)=\int_0^t\Delta^2(u)du$ (4.4.17)

$$dX(t)=\Delta(t)dW(t)+\theta(t)dt$$

$$dX(t)dX(t)$$
$$=\Delta^2(t)dW(t)dW(t)+2\Delta(t)dW(t)\theta(t)dt+\theta^2(t)dtdt$$

Formula for Ito Process

Definition 4.4.5. Let $X(t), t \geq 0$, be an Ito process as described in Definition 4.4.3, and let $\Gamma(t), t \geq 0$, be an adapted process. We define the integral with respect to an Ito process

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Gamma(u)\theta(u)du \quad (4.4.20)$$

Recall the proof of Theorem 4.4.1

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$f\big(T,W(T)\big) = f\big(0,W(0)\big) + \int_0^T f_t\big(t,W(t)\big)\,dt + \int_0^T f_x\big(t,W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt. \quad (4.4.3)$$

Formula for Ito Process

$$
\begin{aligned}
f\big(T, W(T)\big) - f\big(0, W(0)\big) \\
&= \sum_{j=0}^{n-1} \left[f\big(t_{j+1}, W(t_{j+1})\big) - f\big(t_j, W(t_j)\big) \right] \\
&= \sum_{j=0}^{n-1} f_t\big(t_j, W(t_j)\big)(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x\big(t_j, W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{xx}\big(t_j, W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)^2 \\
&\quad + \sum_{j=0}^{n-1} f_{tx}\big(t_j, W(t_j)\big)(t_{j+1} - t_j)\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{tt}\big(t_j, W(t_j)\big)(t_{j+1} - t_j)^2 + \text{ higher-order terms.} \qquad (4.4.9)
\end{aligned}
$$

Formula for Ito Process

$$
\begin{aligned}
&f(T, X(T)) - f(0, X(0)) \\
&= \sum_{j=0}^{n-1} f_t(t_j, X(t_j))(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x(t_j, X(t_j))\big(X(t_{j+1}) - X(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{xx}(t_j, X(t_j))\big(X(t_{j+1}) - X(t_j)\big)^2 \\
&\quad + \sum_{j=0}^{n-1} f_{tx}(t_j, X(t_j))(t_{j+1} - t_j)\big(X(t_{j+1}) - X(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{tt}(t_j, X(t_j))(t_{j+1} - t_j)^2 + \text{ higher-order terms.} \qquad (4.4.21)
\end{aligned}
$$

$\to 0$

$as\ \|\pi\| \to 0$

Recall

$$\lim_{\|\Pi\|\to 0}\left|\sum_{j=0}^{n-1} f_{tx}\big(t_j, W(t_j)\big)(t_{j+1}-t_j)\big(W(t_{j+1})-W(t_j)\big)\right|$$

$$\leq \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tx}\big(t_j, W(t_j)\big)\right|\cdot(t_{j+1}-t_j)\cdot\left|W(t_{j+1})-W(t_j)\right|$$

$$\leq \lim_{\|\Pi\|\to 0}\max_{0\leq k\leq n-1}\left|W(t_{k+1})-W(t_k)\right|\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tx}\big(t_j, W(t_j)\big)\right|(t_{j+1}-t_j)$$

$$= 0\cdot\int_0^T\left|f_{tx}\big(t, W(t)\big)\,dt \;=\; 0. \qquad (4.4.10)$$

The fifth term is treated similarly:

$$\lim_{\|\Pi\|\to 0}\left|\frac{1}{2}\sum_{j=0}^{n-1} f_{tt}\big(t_j, W(t_j)\big)(t_{j+1}-t_j)^2\right|$$

$$\leq \lim_{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}\left|f_{tt}\big(t_j, W(t_j)\big)\right|\cdot(t_{j+1}-t_j)^2$$

$$\leq \frac{1}{2}\lim_{\|\Pi\|\to 0}\max_{0\leq k\leq n-1}(t_{k+1}-t_k)\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tt}\big(t_j, W(t_j)\big)\right|(t_{j+1}-t_j)$$

$$= \frac{1}{2}\cdot 0\cdot\int_0^T f_{tt}\big(t, W(t)\big)dt \;=\; 0. \qquad (4.4.11)$$

The higher-order terms likewise contribute zero to the final answer. □

Formula for Ito Process

$$f(T, X(T)) - f(0, X(0))$$

$$= \sum_{j=0}^{n-1} f_t(t_j, X(t_j))(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x(t_j, X(t_j))\big(X(t_{j+1}) - X(t_j)\big)$$

$$+\frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j, X(t_j))\big(X(t_{j+1}) - X(t_j)\big)^2$$

$$\int_0^T f_x(t, X(t))\, dX(t) = \int_0^T f_x(t, X(t))\Delta(t)\, dW(t) + \int_0^T f_x(t, X(t))\Theta(t)\, dt.$$

$$\frac{1}{2}\int_0^T f_{xx}(t, X(t))\, d[X, X](t) = \frac{1}{2}\int_0^T f_{xx}(t, X(t))\Delta^2(t)\, dt$$

Formula for Ito Process

$$f(T, X(T)) - f(0, X(0))$$

$$= \sum_{j=0}^{n-1} f_t(t_j, X(t_j))(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x(t_j, X(t_j))(X(t_{j+1}) - X(t_j))$$

$$+\frac{1}{2} \sum_{j=0}^{n-1} f_{xx}(t_j, X(t_j))(X(t_{j+1}) - X(t_j))^2$$

$$f(T, X(T))$$

$$= f(0, X(0)) + \int_0^T f_t(t, X(t))\, dt + \int_0^T f_x(t, X(t))\, dX(t)$$

$$+\frac{1}{2} \int_0^T f_{xx}(t, X(t))\, d[X, X](t)$$

$$= f(0, X(0)) + \int_0^T f_t(t, X(t))\, dt + \int_0^T f_x(t, X(t))\Delta(t)\, dW(t)$$

$$+ \int_0^T f_x(t, X(t))\Theta(t)\, dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t))\Delta^2(t)\, dt. \qquad (4.4.22)$$

Section 4.4.3

Let $W(t), t \geq 0$, be a Brownian motion, let $F(t), t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Ito process

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds. \tag{4.4.25}$$

Consider an asset price process given by

$$S(t) = S(0)e^{X(t)} = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}, \tag{4.4.26}$$

where $S(0)$ is nonrandom and positive. We may write $S(t) = f\big(X(t)\big)$, where $f(x) = S(0)e^x$, $f'(x) = S(0)e^x$, and $f''(x) = S(0)e^x$. According to the Itô-Doeblin formula

Section 4.4.3

$$\begin{aligned}
dS(t) &= df\big(X(t)\big) \\
&= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t) \\
&= S(0)e^{X(t)}\,dX(t) + \frac{1}{2}S(0)e^{X(t)}\,dX(t)\,dX(t) \\
&= S(t)\,dX(t) + \frac{1}{2}S(t)\,dX(t)\,dX(t) \\
&= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t). \qquad (4.4.27)
\end{aligned}$$

$$dX(t) = \sigma(t)\,dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt,$$

$$dX(t)\,dX(t) = \sigma^2(t)\,dW(t)\,dW(t) = \sigma^2(t)\,dt.$$

Section 4.4.3

$\alpha = 0$ $$dS(t) = \sigma(t)S(t)\,dW(t).$$

integration $$S(t) = S(0) + \int_0^t \sigma(s)S(s)\,dW(s).$$

martingale $$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) - \frac{1}{2}\int_0^t \sigma^2(s)\,ds\right\} \quad (4.4.29)$$

Section 4.4.3

Theorem 4.4.9 (Ito integral of a deterministic integrand)

Let $W(s), s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s) dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s) ds$.

Proof

$I(t)$ is martingale and $I(0)=0$ ➔ mean=0

$$\mathrm{Var} I(t) = \mathbb{E} I^2(t) = \int_0^t \Delta^2(s)\, ds.$$ by Theorem 4.3.1 (Ito's isometry)

Section 4.4.3

Normal distribution

$$\mathbb{E}e^{uI(t)} = \exp\left\{\frac{1}{2}u^2 \int_0^t \Delta^2(s)\,ds\right\} \text{ for all } u \in \mathbb{R}. \tag{4.4.30}$$

$$\mathbb{E}\exp\left\{uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)\,ds\right\} = 1,$$

$$\mathbb{E}\exp\left\{\int_0^t u\Delta(s)\,dW(s) - \frac{1}{2}\int_0^t \left(u\Delta(s)\right)^2 ds\right\} = 1. \tag{4.4.31}$$

Martingale, it is a generalized geometric Brownian motion with mean rate of return $\alpha = 0$, $\sigma(s) = u\Delta(s)$

$t = 0$,value$= 1$