3.7 REFLECTION PRINCIPLE

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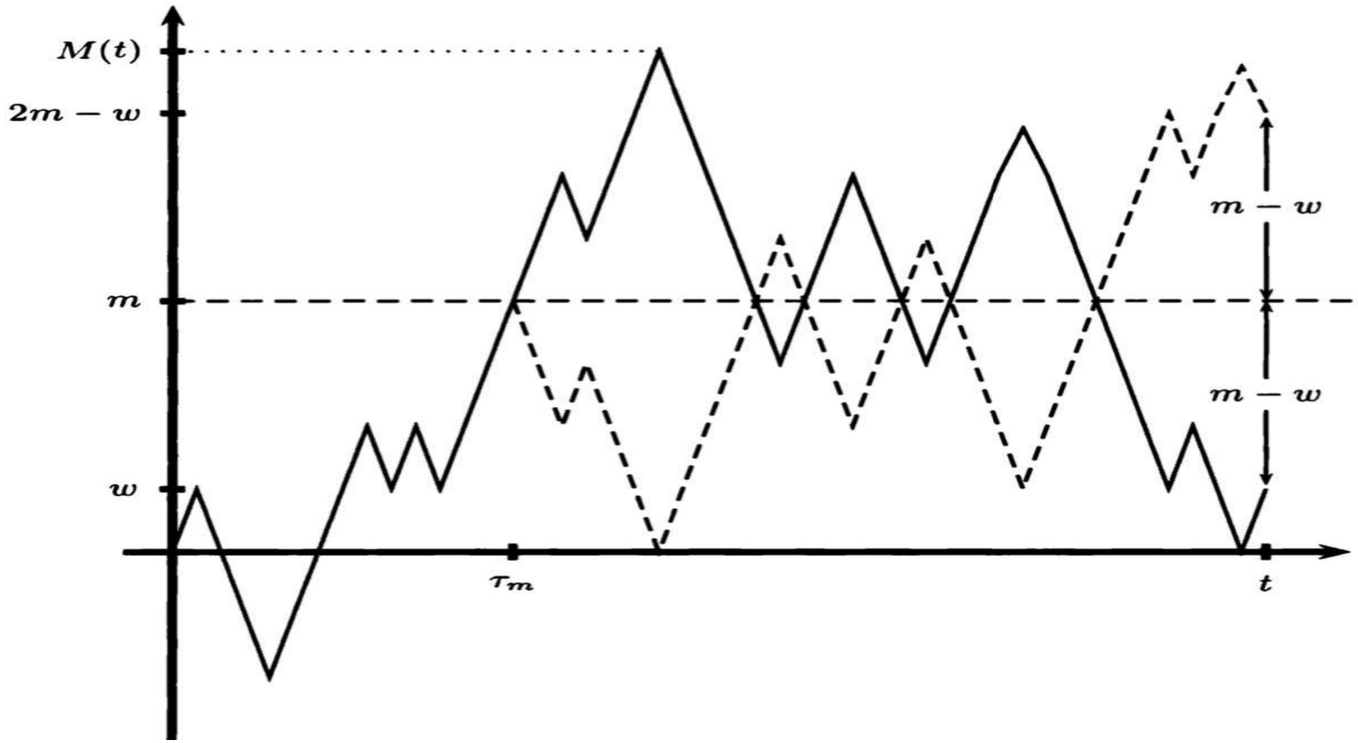
- 3.7.1 Reflection Equality
- 3.7.2 First Passage Time Distribution
- 3.7.3 Distribution of Brownian Motion and Its Maximum

3.7.1 Reflection Equality



$$p\{\tau_m \leq t, W(t) \leq w\} = p\{W(t) \geq 2m - w\}, \quad w \leq m, m > 0 \qquad (3.7.1)$$

3.7.2 First Passage Time Distribution

- Thm 3.7.1

For all m$\neq 0$, the random variable τ_m has cumulative distribution function

$$p\{\tau_m \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, \qquad t \geq 0$$

(3.7.2)

And density

$$f_{\tau_m}(t) = \frac{d}{dt} p\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, \qquad t \geq 0$$

(3.7.3)

- # Proof of Thm 3.7.1

- We first consider the case m > 0.We substitute w = m into the reflection formula (3.7.1) to obtain

$$p\{\tau_m \le t, W(t) \le w\}=p\{\tau_m \le t, W(t) \le m\} = p\{W(t) \ge m\}$$

If $W(t) \ge m$, then we guaranteed that $\tau_m \le t$, so

$$p\{\tau_m \le t, W(t) \ge w\} = p\{\tau_m \le t, W(t) \ge m\} = p\{W(t) \ge m\}$$

We obtain

$$p\{\tau_m \le t\} = p\{\tau_m \le t, W(t) \le m\} + p\{\tau_m \le t, W(t) \ge m\}$$

$$= 2p\{W(t) \ge m\} = \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}}\, dx$$

$$p\{\tau_m \le t\} = p\{\tau_m \le t, W(t) \le m\} + p\{\tau_m \le t, W(t) \ge m\}$$

$$= 2p\{W(t) \ge m\}$$

$$= \frac{2}{\sqrt{2\pi t}} \int_m^{\infty} e^{-\frac{x^2}{2t}}\, dx$$

Let $y = \frac{x}{\sqrt{t}}$,**dy=**$\frac{dx}{\sqrt{t}}$

$$= \frac{2}{\sqrt{2\pi}} \int_{\frac{m}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{y^2}{2}} dy$$

$$= \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}\, dy, \qquad t \ge 0$$

leads to (3.7.2)

- If m <0,

τ_m and $\tau_{|m|}$ have the same distribution, and (3.7.2) provides the c.d.f of the latter.

$$p\{\tau_m \leq t\}=p\{\tau_{|m|} \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, \qquad t \geq 0 \qquad (3.7.2)$$

- Finally, (3.7.3) is obtained by dirrerentiating (3.7.2) with respect to t.

- $f_{\tau_m}(t) = \frac{d}{dt} p\{\tau_m \leq t\} = \frac{d}{dt}\left\{\frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy\right\}$

$$= \frac{d}{dy} \cdot \frac{dy}{dt}\left\{\frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy\right\}$$

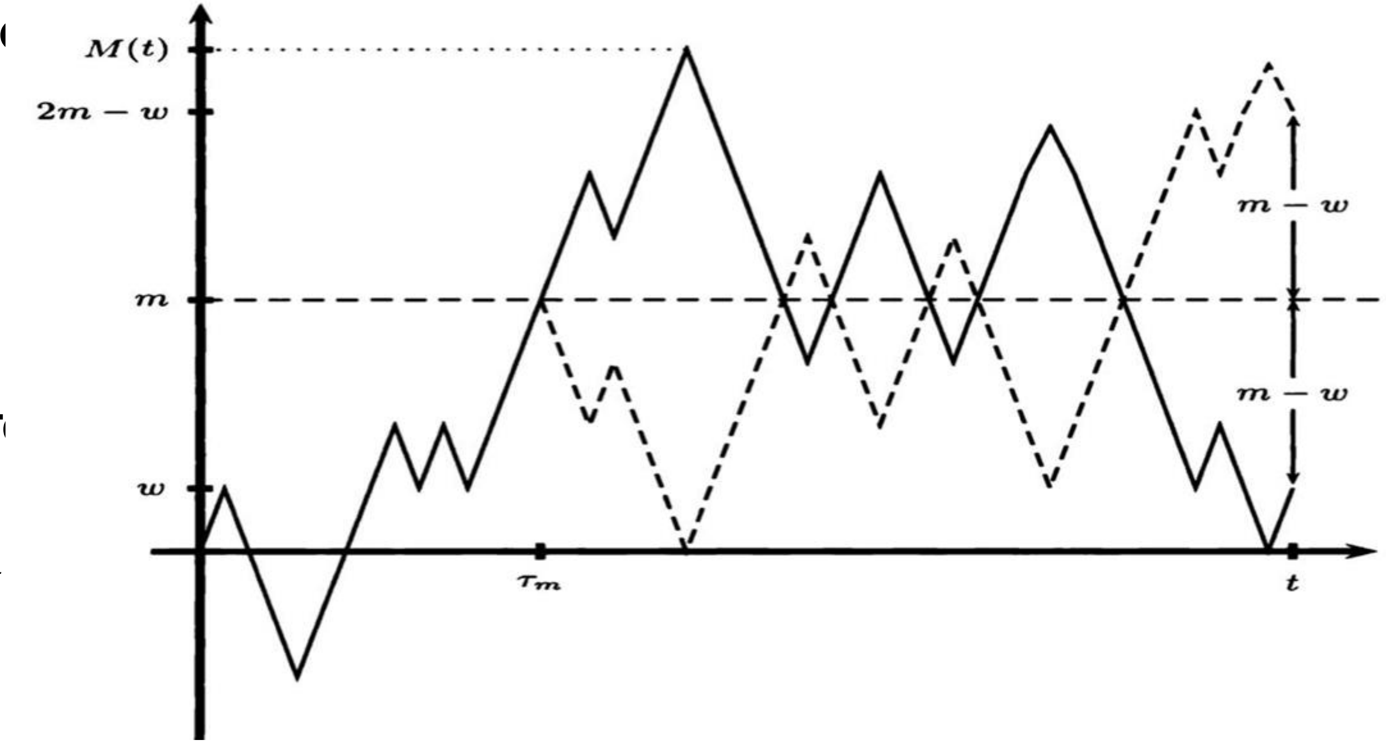
$$= \frac{dy}{dt}\left\{-\frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right\} \text{ ,代入 } \mathbf{y} = \frac{|m|}{\sqrt{t}}$$

$$= -\frac{2}{\sqrt{2\pi}} e^{-\frac{m^2}{2t}} \cdot -\frac{|m|}{2t\sqrt{t}}$$

$$= \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, \qquad t \geq 0$$

3.7.3 Distribution of Brownian Motion and Its Maximum

- We defin



(3.7.5)

For positiv

$p\{M$ $> 0.$

(3.7.6)

The joint distribution of W(t) and M(t)

- Thm 3.7.3

  For $t > 0$, the joint density of $(M(t), W(t))$ is

$$f_{M(t),W(t)}(m,\omega) = \frac{2(2m-\omega)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \omega \le m, m > 0.$$

(3.7.7)

- Proof

Because

$$p\{M(t) \geq m, W(t) \leq w\} = \int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)dydx$$

and

$$p\{W(t) \geq 2m - w\} = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}} dz$$

Since $p\{\tau_m \leq t, W(t) \leq w\} = p\{W(t) \geq 2m - w\}$, $\text{w} \leq m, m > 0$ (3.7.1)

We have

$$\int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}} dz$$

$$\int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}} dz$$

We differentiate first with respect to m to obtain

$$-\int_{-\infty}^w f_{M(t),W(t)}(x,y)dy = -\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

We next differentiate with respect to w to see that

$$-f_{M(t),W(t)}(x,y)dy = -\frac{2(2m-w)}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

$$f_{M(t),W(t)}(x,y)dy = \frac{2(2m-w)}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

Corollary 3.7.4

- The conditional distribution of M(t) given is W(t) = w

$$f_{M(t)|w(t)}(m|w) = \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}, \qquad w \le m, m > 0.$$

Proof

$$f_{M(t)|W(t)}(m|w) = \frac{f_{M(t),W(t)}(m,w)}{f_{w(t)}(w)}$$

$$= \left(\frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\right) / \left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}}\right)$$

$$= \frac{2(2m-w)}{t\sqrt{2\pi t}} \cdot \sqrt{2\pi t} \cdot e^{-\frac{(2m-w)^2}{2t}+\frac{w^2}{2t}}$$

$$= \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}$$

$$f_{M(t),W(t)}(x,y) = \frac{2(2m-w)}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

$$f_{w(t)}(w) \sim N(0,1) \qquad (3.7.7)$$

Thank you for listening~