ODE Teaching Part 2 for Fall, 2016

Revised on 2016-12-23

Material from the Instructor (based on Logan (3rd edition) and Boyce-DiPrima's book (10th edition))

Chapter 4: Linear Systems of Equations

0.1 Linear system in $\mathbb{R}^n$ with constant coefficients

Definition 0.1 *Let A be an $n \times n$ real matrix. The system of equation*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x} = \mathbf{x}(t) = (x_1(t), \ \dots, x_n(t))^T \in \mathbb{R}^n \tag{1}$$

is called a ***first order*** $n \times n$ ***linear system of ODE with constant coefficients*** *(since A is a constant matrix).*

Remark 0.2 *If there is no confusion, we will just write $\mathbf{x}(t) = (x_1(t), \ \dots, x_n(t))^T$ as $\mathbf{x}(t) = (x_1(t), \ \dots, x_n(t))$.*

Remark 0.3 *For a given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$, we have existence and uniqueness theorem for (1). Also, any solution is defined on $t \in (-\infty, \infty)$.*

Example 0.4 *Consider the 2×2 linear system of equations with constant coefficients*

$$\begin{cases} x_1'(t) = 3x_1 - 4x_2 \\ x_2'(t) = -x_1 + 5x_2. \end{cases}$$

One can write it as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t)), \quad \text{where} \quad A = \begin{pmatrix} 3 & -4 \\ -1 & 5 \end{pmatrix}.$$

Example 0.5 *Consider the second order linear equation*

$$x''(t) + x(t) = 0, \quad t \in (-\infty, \infty). \tag{2}$$

We know that its general solution is given by

$$x(t) = c_1 \cos t + c_2 \sin t, \quad t \in (-\infty, \infty), \quad c_1, \ c_2 \text{ are constants.}$$

If we let $y(t) = x'(t)$ (view y as a new variable), (2) gives

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x,$$

i.e. the vector-valued function $\mathbf{x}(t) = (x(t), y(t))$ satisfies the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x}, \quad i.e. \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

One can check that (2) is equivalent *to (3). The same observation applies to higher order linear equations with constant coefficients. The upshot is that a n-th order linear equation with constant coefficients is equivalent to a first order $n \times n$ linear system of ODE with constant coefficients.*

Lemma 0.6 *If* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are both solutions to (1) on some interval* I*, then their linear combination*

$$\mathbf{z}(t) = c_1\mathbf{x}(t) + c_2\mathbf{y}(t), \quad t \in I$$

is also a solution of (1) on I*. Here* $c_1,\ c_2$ *are arbitrary constants.*

Remark 0.7 *This says that the solution space of (1) has the structure of a* ***vector space****.*

Proof. This is obvious. □

We first need some results from linear algebra:

Lemma 0.8 *If an* $n \times n$ *matrix* A *has* n ***distinct real*** eigenvalues $\lambda_1,\ ...,\ \lambda_n$ *with corresponding eigenvectors* $v_1,\ ...,\ v_n$*, then* $v_1,\ ...,\ v_n$ *are linearly independent in* $\mathbb{R}^n$*.*

Proof. We first claim that v_1 and v_2 are independent. Otherwise, we would have $v_1 = cv_2$ for some constant $c \neq 0$. Hence we get (applying A onto it) $\lambda_1 v_1 = c\lambda_2 v_2$. But we also have $\lambda_1 v_1 = c\lambda_1 v_2$ and so $c\lambda_2 v_2 = c\lambda_1 v_2$. This will force $\lambda_1 = \lambda_2$, impossible. Hence v_1 and v_2 are independent. Similarly if we have $v_3 = \alpha v_1 + \beta v_2$ with $\alpha^2 + \beta^2 \neq 0$, then

$$\begin{cases} \lambda_3 v_3 = \alpha\lambda_1 v_1 + \beta\lambda_2 v_2 \\ \lambda_3 v_3 = \alpha\lambda_3 v_1 + \beta\lambda_3 v_2 \end{cases}$$

which implies $\alpha(\lambda_1 - \lambda_3) v_1 + \beta(\lambda_2 - \lambda_3) v_2 = 0$ and so $\alpha = \beta = 0$, a contradiction. Thus $v_1,\ v_2,\ v_3$ are independent. Keep going. One can see that $v_1,\ ...,\ v_n$ are linearly independent. □

Lemma 0.9 *If an* $n \times n$ *matrix* A *has* n ***distinct real*** eigenvalues $\lambda_1,\ ...,\ \lambda_n$ *with corresponding eigenvectors* $v_1,\ ...,\ v_n$*, then*

$$P^{-1}AP = diag(\lambda_1,\ ...,\ \lambda_n), \tag{4}$$

where $P = (v_1,\ ...,\ v_n)$ *(each* v_i *is a* ***column*** *eigenvector). Here* $diag(\lambda_1,\ ...,\ \lambda_n)$ *means the diagonal matrix with diagonal elements* $\lambda_1,\ ...,\ \lambda_n$*.*

Proof. Note that

$$AP = P \times diag(\lambda_1,\ ...,\ \lambda_n)$$

and the proof is done. □

Remark 0.10 *Compare the difference between* $P \times diag(\lambda_1,\ ...,\ \lambda_n)$ *(the i-th* column *of* P *is multiplied by* λ_i*) and* $diag(\lambda_1,\ ...,\ \lambda_n) P$ *(the i-th* row *of* P *is multiplied by* λ_i*).*

Lemma 0.11 *If* λ *is a* ***real*** eigenvalue *of* A *with corresponding* eigenvector $v \in \mathbb{R}^n$ *(note that* $v \neq 0$*), then the function*

$$\mathbf{x}(t) = e^{\lambda t} v, \quad t \in (-\infty, \infty)$$

is a solution of (1) on $(-\infty, \infty)$*.*

Proof. We have $Av = \lambda v$. Hence

$$\frac{d\mathbf{x}}{dt}(t) = \lambda e^{\lambda t} v = A\left(e^{\lambda t} v\right) = A\mathbf{x}(t).$$

□

Lemma 0.12 ***(First version.)*** *If A has n* ***distinct real*** *eigenvalues $\lambda_1, \ ..., \ \lambda_n$ with corresponding eigenvectors $v_1, \ ..., \ v_n$, then if $\mathbf{x}(t) \in \mathbb{R}^n$ is a solution of (1) on $(-\infty, \infty)$, it can be expressed as*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n, \quad t \in (-\infty, \infty) \tag{5}$$

for some constants $c_1, \ ..., \ c_n$. Therefore, the ***general solution*** *of the linear system $d\mathbf{x}/dt = A\mathbf{x}$ in this case (i.e., A has n* ***distinct real*** *eigenvalues) is given by (5).*

Proof. At any time $t \in (-\infty, \infty)$ one can decompose $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = a_1(t) v_1 + \cdots + a_n(t) v_n$$

for some coefficient functions $a_1(t), \ ..., \ a_n(t)$. We now have

$$\frac{d\mathbf{x}}{dt}(t) = a_1'(t) v_1 + \cdots + a_n'(t) v_n = A\mathbf{x}(t) = \lambda_1 a_1(t) v_1 + \cdots + \lambda_n a_n(t) v_n.$$

This implies $a_1'(t) = \lambda_1 a_1(t), \ ..., \ a_n'(t) = \lambda_n a_n(t)$. Hence there exist constants $c_1, \ ..., \ c_n$ such that

$$a_1(t) = c_1 e^{\lambda_1 t}, \quad ...., \quad a_n(t) = c_n e^{\lambda_n t}, \quad t \in (-\infty, \infty).$$

The proof is done. □

Remark 0.13 ***(Matrix representation of the solution.)*** *One can express (5) as*

$$\mathbf{x}(t) = PD(t)C, \quad t \in (-\infty, \infty) \tag{6}$$

where $P = (v_1, \ ..., \ v_n)$ (each v_i is a column eigenvector of λ_i), $D = diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$, and C is an arbitrary constant (column) vector.

Remark 0.14 ***(Matrix representation of the solution.)*** *In case there is a initial condition $\mathbf{x}(0) = \mathbf{x}_0$, then one just solve for $C = (c_1, \ ..., \ c_n)$ so that*

$$c_1 v_1 + \cdots + c_n v_n = \mathbf{x}_0. \tag{7}$$

In matrix form we have $PC = \mathbf{x}_0$ (column vector), where $P = (v_1, \ ..., \ v_n)$ (each v_i is a column eigenvector of λ_i) and $C = (c_1, ...c_n)$ (column vector) is to be solved. Hence we get $C = P^{-1}\mathbf{x}_0$ and so

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n \\ &= PD(t) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = PD(t) P^{-1}\mathbf{x}_0, \qquad \mathbf{x}(0) = \mathbf{x}_0, \end{aligned} \tag{8}$$

where $D(t)$ is the diagonal matrix $diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$. Since the solution for $c_1, \ ..., \ c_n$ in the equation (7) is unique, we know that there is a unique solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{9}$$

and the solution is defined on $t \in (-\infty, \infty)$.

We can summarize the conclusion in the above remark as:

Lemma 0.15 *(**Second version.**) Assume A has n **distinct real** eigenvalues $\lambda_1, \ ..., \ \lambda_n$ with corresponding eigenvectors $v_1, \ ..., \ v_n$ and let $P = (v_1, \ ..., \ v_n)$. Then the **general solution** to the equation $d\mathbf{x}/\mathbf{dt} = A\mathbf{x}$ is given by*

$$\mathbf{x}(t) = PD(t)C, \quad \textit{where } C \textit{ is an arbitrary constant vector,} \quad t \in (-\infty, \infty) \tag{10}$$

*and the **unique solution** to the initial value problem (9) is given by*

$$\mathbf{x}(t) = PD(t)P^{-1}\mathbf{x}_0, \quad t \in (-\infty, \infty) \tag{11}$$

where $D(t)$ is the diagonal matrix $diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$.

Remark 0.16 *(**Important.**) If we choose different eigenvectors $w_1, \ ..., \ w_n$ for $\lambda_1, \ ..., \ \lambda_n$, then there are numbers $a_1, \ ..., \ a_n$ (all are nonzero) so that*

$$w_1 = a_1 v_1, \qquad ...., \qquad w_n = a_n v_n.$$

*Hence the matrix $Q = (w_1, \ ..., \ w_n)$ (each w_i is a **column** eigenvector) satisfies the identity*

$$Q = PM, \quad \textit{where} \quad M = diag(a_1, \ ..., \ a_n)$$

and so

$$QD(t)Q^{-1}\mathbf{x}_0 = PMD(t)(PM)^{-1}\mathbf{x}_0 = P\left[MD(t)M^{-1}\right]P^{-1}\mathbf{x}_0 = PD(t)P^{-1}\mathbf{x}_0.$$

Therefore, the solution formula (11) is independent of the choice of eigenvectors for the eigenvalues $\lambda_1, \ ..., \ \lambda_n$.

Proof. The proof is already done due to Remark 0.13 and Remark 0.14. Here we shall give a different proof revealing the importance of eigenvalues and eigenvectors. Suppose we want to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{12}$$

where the n variables $x_1, \ ...x_n$ are coupled in each equation of the system. The idea is to **decouple** the variables $x_1, \ ...x_n$ by a linear change of variables. Let $\mathbf{x}(t) = P\mathbf{y}(t)$, where P is some constant *nonsingular* $n \times n$ matrix and $\mathbf{y}(t) = (y_1(t), \ ... \ , y_n(t))$ is the new variable. If we plug $\mathbf{x}(t) = P\mathbf{y}(t)$ into (12), we get

$$P\frac{d\mathbf{y}}{dt} = AP\mathbf{y}, \quad P\mathbf{y}(0) = \mathbf{x}_0.$$

Hence the new equation for the new variable $\mathbf{y}(t)$ is

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y}, \quad \mathbf{y}(0) = P^{-1}\mathbf{x}_0.$$

Therefore, if $P^{-1}AP$ is a diagonal matrix $diag(\lambda_1, \ ..., \ \lambda_n)$ (in such a case, $\lambda_1, \ ..., \ \lambda_n$ must be eigenvalues of A and the column vectors $v_1, \ ..., \ v_n$ of P must be eigenvectors of A) we will have

$$\frac{dy_1}{dt} = \lambda_1 y_1, \quad \frac{dy_2}{dt} = \lambda_2 y_2, \quad \cdots, \quad \frac{dy_n}{dt} = \lambda_n y_n, \quad \mathbf{y}(0) = P^{-1}\mathbf{x}_0$$

and the solution $\mathbf{y}(t)$ is (**note that now the system has been decoupled**)

$$\mathbf{y}(t) = diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)\mathbf{y}(0) = diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)\left(P^{-1}\mathbf{x}_0\right) = D(t)P^{-1}\mathbf{x}_0.$$

Thus

$$\mathbf{x}(t) = P\mathbf{y}(t) = PD(t)P^{-1}\mathbf{x}_0$$

and the proof is done. □

Example 0.17 *Consider the linear system*

$$\begin{cases} x_1'(t) = 3x_1 - x_2 \\ x_2'(t) = 4x_1 - 2x_2. \end{cases} \tag{13}$$

We have

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

and $\lambda_1 = 2,\ \lambda_2 = -1,\ v_1 = (1,1),\ v_2 = (1,4).$ *Thus*

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x}_0 \\ &= \frac{1}{3}\begin{pmatrix} 4e^{2t} - e^{-t} & -e^{2t} + e^{-t} \\ 4e^{2t} - 4e^{-t} & -e^{2t} + 4e^{-t} \end{pmatrix}\mathbf{x}_0 \end{aligned}$$

is the solution of (13) with initial data $\mathbf{x}_0$. *One can also use the formula*

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

and solve for $c_1,\ c_2$ *satisfying the system*

$$\mathbf{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \mathbf{x}_0.$$

In general, the $n \times n$ real matrix A may have *repeated* or *complex* eigenvalues. To discuss the general solution of the linear system $d\mathbf{x}/dt = A\mathbf{x}$, we need to introduce the following concept of **the exponential of a real matrix** A :

Definition 0.18 *Let A be an $n \times n$ real matrix. We define its **exponential** e^A to be the $n \times n$ real matrix*

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{14}$$

Remark 0.19 *By definition, we have $e^0 = I$, where 0 is the zero $n \times n$ matrix.*

Remark 0.20 *The definition is motivated by the Taylor series for e^x :*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in (-\infty, \infty).$$

Example 0.21 *If $A = diag(\lambda_1,\ ...,\ \lambda_n)$, then $e^A = diag\left(e^{\lambda_1},\ ...,\ e^{\lambda_n}\right)$.*

Of course, we need to check the following:

Lemma 0.22 *Let $A = (a_{ij})$ be an $n \times n$ real matrix. Then the series*

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{15}$$

converges absolutely. *In particular, the above series converges and is a well-defined matrix, denoted as e^A.*

Proof. Let $M = \max_{1\le i,j\le n} |a_{ij}|$ and for convenience, look at the $(1,1)$ term $a_{11}^{(k)}$ in each A^k. We have

$$\left|a_{11}^{(1)}\right| \le M, \quad \left|a_{11}^{(2)}\right| \le nM^2, \quad \left|a_{11}^{(3)}\right| \le n^2M^3, \quad \left|a_{11}^{(4)}\right| \le n^3M^4, \quad ..., \quad etc.$$

Hence the series at the $(1,1)$ position in e^{tA}, which is

$$1 + a_{11}^{(1)} + \frac{a_{11}^{(2)}}{2!} + \frac{a_{11}^{(3)}}{3!} + \cdots,$$

satisfies

$$\begin{aligned} &1 + \left|a_{11}^{(1)}\right| + \frac{\left|a_{11}^{(2)}\right|}{2!} + \frac{\left|a_{11}^{(3)}\right|}{3!} + \frac{\left|a_{11}^{(4)}\right|}{4!} + \cdots \\ &\le 1 + M + \frac{nM^2}{2!} + \frac{n^2M^3}{3!} + \frac{n^3M^4}{4!} + \cdots \le e^{nM}. \end{aligned}$$

That is, the partial sum of the *positive* series

$$1 + \left|a_{11}^{(1)}\right| + \frac{\left|a_{11}^{(2)}\right|}{2!} + \frac{\left|a_{11}^{(3)}\right|}{3!} + \frac{\left|a_{11}^{(4)}\right|}{4!} + \cdots$$

has upper bound. Hence it must converge. The same argument applies to other components and the proof is done. □

Lemma 0.23 *Let A, B be two $n \times n$ real matrices such that $B = P^{-1}AP$ (in such a case we say B is* ***similar*** *to A), where P is an invertible $n \times n$ matrix. Then*

$$e^B = P^{-1}e^AP. \tag{16}$$

In particular, if $B = diag\,(\lambda_1, \ ..., \ \lambda_n)$, then $e^A = P diag\left(e^{\lambda_1}, \ ..., \ e^{\lambda_n}\right)P^{-1}$.

Remark 0.24 *(**Important.**) The above says that, to compute e^A, it suffices to diagonalize the matrix A (if this can be done).*

Proof. By definition we have

$$\begin{aligned} e^B = e^{P^{-1}AP} &= I + P^{-1}AP + \frac{\left(P^{-1}AP\right)^2}{2!} + \frac{\left(P^{-1}AP\right)^3}{3!} + \cdots \\ &= P^{-1}IP + P^{-1}AP + \frac{\left(P^{-1}AP\right)^2}{2!} + \frac{\left(P^{-1}AP\right)^3}{3!} + \cdots \\ &= P^{-1}\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)P, \end{aligned}$$

which means that the n-th partial sum (denote it as s_n) in the series for e^B is given by

$$s_n = P^{-1}\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!}\right)P.$$

Since we have

$$\lim_{n\to\infty}\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!}\right) = e^A,$$

we get

$$\lim_{n\to\infty} s_n = P^{-1}e^AP.$$

The proof is done. □

Lemma 0.25 *If $\lambda \in \mathbb{R}$ is an eigenvalue of an $n \times n$ real matrix A with corresponding eigenvector $v \neq 0 \in \mathbb{R}^n$, then $e^A v = e^\lambda v$.*

Proof. We have

$$\begin{aligned} e^A v &= \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right) v \\ &= Iv + Av + \frac{A^2}{2!}v + \frac{A^3}{3!}v + \cdots = v + \lambda v + \frac{\lambda^2}{2!}v + \frac{\lambda^3}{3!}v + \cdots \\ &= \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right) v = e^\lambda v. \end{aligned}$$

□

Lemma 0.26 *If B is an $n \times n$ real matrix satisfying $AB = BA$, then we have*

$$Be^A = e^A B \tag{17}$$

and

$$e^{A+B} = e^A e^B = e^B e^A. \tag{18}$$

In particular, for any $n \times n$ real matrix A the matrix e^A is always invertible with

$$\left(e^A\right)^{-1} = e^{-A}. \tag{19}$$

Remark 0.27 ***(Interesting)*** *The condition $AB = BA$ in (17) and (18) are necessary. There exist two 2×2 real matrices $A,\ B$ such that $AB \neq BA$ and*

$$e^A e^B \neq e^B e^A.$$

For example, take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^B = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \quad e^A e^B = \begin{pmatrix} e & 1 \\ 0 & 1 \end{pmatrix}, \quad e^B e^A = \begin{pmatrix} e & e \\ 0 & 1 \end{pmatrix}$$

and

$$e^{A+B} = e^{B+A} = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}.$$

Thus $e^{A+B} = e^{B+A}$, $e^A e^B$ and $e^B e^A$ are all different.

Proof. (**Omit in class. See Remark 0.32 also.**) For (17), we have

$$\begin{aligned} Be^A &= B\left(\lim_{j\to\infty} s_j\right), \quad s_j = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^j}{j!} \\ &= \lim_{j\to\infty} (Bs_j) = \lim_{j\to\infty} (s_j B) = \left(\lim_{j\to\infty} s_j\right) B = e^A B. \end{aligned}$$

The proof of (18) is more delicate. For convenience we look at the case $n = 2$. For given $\varepsilon > 0$, we can write e^A as

$$e^A = \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^j}{j!}\right) + \cdots = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix},$$

where each term $*$ in the second matrix satisfies $|*| < \varepsilon$ (if j is large enough). Similar we have

$$e^B = \left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^j}{j!}\right) + \cdots = \begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix},$$

where $|*| < \varepsilon$. Now

$$\begin{aligned} e^A e^B &= \left[\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix}\right]\left[\begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix}\right] \\ &= \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{aligned}$$

and (here we use the assumption $AB = BA$)

$$\begin{aligned} &\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix} \\ &= \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^j}{j!}\right)\left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^j}{j!}\right) \\ &= I + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \cdots \\ &\text{(this is not same as } I + (A+B) + \frac{(A+B)^2}{2!} + \cdots + \frac{(A+B)^j}{j!}\text{).} \end{aligned}$$

However, in the limit we can get

$$\lim_{j\to\infty}\left(\begin{array}{c} \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^j}{j!}\right)\left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^j}{j!}\right) \\ -\left(I + (A+B) + \frac{(A+B)^2}{2!} + \cdots + \frac{(A+B)^j}{j!}\right) \end{array}\right) = 0,$$

which implies

$$\lim_{j\to\infty}\left(\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ \tilde{c}_j & \tilde{d}_j \end{pmatrix}\right) = e^{A+B}$$

and then

$$e^A e^B = e^{A+B}$$

The proof is done.

To prove the last identity, note that A and $-A$ are commutable, which implies

$$e^A e^{-A} = e^{A+(-A)} = e^0 = I \quad \text{(similarly, we have } e^{-A}e^A = I\text{)}.$$

Therefore $\left(e^A\right)^{-1} = e^{-A}$. □

Lemma 0.28 *Let $A = (a_{ij})$ be an $n \times n$ real matrix and let $I \subset \mathbb{R}$ be a* bounded *interval. Then the series*

$$e^{tA} := I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \tag{20}$$

converges absolutely and ***uniformly*** *for all $t \in I$. In particular, e^{tA} is defined on $t \in (-\infty, \infty)$.*

Proof. Since the interval I is bounded, each component of the matrix $tA = (ta_{ij})$ is bounded. Let $M = \max_{t\in I, 1\le i,j\le n} |ta_{ij}|$ and for convenience, look at the $(1,1)$ term $a_{11}^{(k)}(t)$ in each $(tA)^k$. We have

$$\left|a_{11}^{(1)}(t)\right| \le M, \quad \left|a_{11}^{(2)}(t)\right| \le nM^2, \quad \left|a_{11}^{(3)}(t)\right| \le n^2M^3, \quad \left|a_{11}^{(4)}(t)\right| \le n^3M^4, \quad ..., \quad etc.$$

Hence the series at the $(1,1)$ position in e^{tA}, which is

$$1+a_{11}^{(1)}(t)+\frac{a_{11}^{(2)}(t)}{2!}+\frac{a_{11}^{(3)}(t)}{3!}+\cdots,\quad t\in I \tag{21}$$

satisfies

$$\begin{aligned}&1+\left|a_{11}^{(1)}(t)\right|+\frac{\left|a_{11}^{(2)}(t)\right|}{2!}+\frac{\left|a_{11}^{(3)}(t)\right|}{3!}+\frac{\left|a_{11}^{(4)}(t)\right|}{4!}+\cdots\\ &\le 1+M+\frac{nM^2}{2!}+\frac{n^2M^3}{3!}+\frac{n^3M^4}{4!}+\cdots\le e^{nM}\quad\text{for all}\quad t\in I.\end{aligned}$$

By the Weierstrass M-test in Advanced Calculus (see Rudin, p. 148), the series (21) converges uniformly on I. The same argument applies to other components and the proof is done. □

As a consequence of Lemma 0.28, we have:

Lemma 0.29 *Let $A=(a_{ij})$ be an $n\times n$ real matrix. We have*

$$\frac{d}{dt}e^{tA}=Ae^{tA}=e^{tA}A\quad\textit{for all}\quad t\in(-\infty,\infty). \tag{22}$$

Proof. We already know that the series

$$s_j(t)=I+tA+\frac{t^2A^2}{2!}+\frac{t^3A^3}{3!}+\cdots+\frac{t^jA^j}{j!}$$

converges uniformly (as $j\to\infty$) on any finite interval $t\in(a,b)$ to e^{tA}. Similarly the series

$$\frac{d}{dt}s_j(t)=A+\frac{2tA^2}{2!}+\frac{3t^2A^3}{3!}+\cdots+\frac{jt^{j-1}A^j}{j!}=A\left(I+tA+\frac{t^2A^2}{2!}+\cdots+\frac{t^{j-1}A^{j-1}}{(j-1)!}\right) \tag{23}$$

also converges uniformly (as $j\to\infty$) on $t\in(a,b)$ to Ae^{tA}. Hence (see Rudin's book "Principles of Mathematical Analysis", p. 152, Theorem 7.17) one can change the order of differentiation and limit, and get

$$\begin{aligned}\frac{d}{dt}e^{tA}&=\frac{d}{dt}\left(\lim_{j\to\infty}s_j(t)\right)=\lim_{j\to\infty}\left(\frac{d}{dt}s_j(t)\right)\\ &=\lim_{j\to\infty}\left[A\left(I+tA+\frac{t^2A^2}{2!}+\cdots+\frac{t^{j-1}A^{j-1}}{(j-1)!}\right)\right]=Ae^{tA}\quad\text{for all}\quad t\in(a,b).\end{aligned}$$

Since the interval (a,b) can be arbitrary, the identity is valid for all $t\in(-\infty,\infty)$. This proves the first identity. For the second identity, we can also write (23) as

$$\frac{d}{dt}s_j(t)=\left(I+tA+\frac{t^2A^2}{2!}+\cdots+\frac{t^{j-1}A^{j-1}}{(j-1)!}\right)A$$

and get $\frac{d}{dt}e^{tA}=e^{tA}A$, $t\in(-\infty,\infty)$. One can also use the fact that A and tA commute and by Lemma 0.26 we obtain $Ae^{tA}=e^{tA}A$ for all $t\in(-\infty,\infty)$. □

We are ready to state the following fundamental theorem for a linear system:

Theorem 0.30 *(**Fundamental theorem for linear system.**) For a given $\mathbf{x}_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{24}$$

has a unique solution $\mathbf{x}(t) \in \mathbb{R}^n$ defined on $(-\infty, \infty)$ and is given by

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0, \quad t \in (-\infty, \infty). \tag{25}$$

*Here e^{tA} is the **exponential matrix** of the matrix tA.*

Remark 0.31 *In case A is diagonalizable with $P^{-1}AP = diag(\lambda_1, \ ..., \ \lambda_n)$ (call it D), we have*

$$e^{tA}\mathbf{x}_0 = e^{t\left(PDP^{-1}\right)}\mathbf{x}_0 = e^{P(tD)P^{-1}}\mathbf{x}_0 = Pe^{tD}P^{-1}\mathbf{x}_0 = PD(t)P^{-1}\mathbf{x}_0, \tag{26}$$

where $D(t)$ is the diagonal matrix $diag\left(e^{\lambda_1 t}, \ ..., \ e^{\lambda_n t}\right)$. This is the same as (11).

Proof. We first have $\mathbf{x}(0) = e^0\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$ and

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(e^{tA}\mathbf{x}_0\right) = \left(\frac{d}{dt}e^{tA}\right)\mathbf{x}_0 = \left(Ae^{tA}\right)\mathbf{x}_0 = A\left(e^{tA}\mathbf{x}_0\right) = A\mathbf{x}(t), \quad \forall\, t \in (-\infty, \infty).$$

Thus it is a solution of (24) on $(-\infty, \infty)$.

If $\mathbf{y}(t)$ is another solution on some interval I, $0 \in I$, with $\mathbf{y}(0) = \mathbf{x}_0$, we look at

$$\begin{aligned} \frac{d}{dt}\left(e^{-tA}\mathbf{y}(t)\right) &= \frac{d}{dt}\left(e^{t(-A)}\mathbf{y}(t)\right) = (-A)e^{t(-A)}\mathbf{y}(t) + e^{t(-A)}\frac{d\mathbf{y}}{dt}(t) \\ &= (-A)e^{t(-A)}\mathbf{y}(t) + e^{t(-A)}A\mathbf{y}(t) = 0, \quad \forall\, t \in I, \end{aligned}$$

where we have used the identity $Ae^{t(-A)} = e^{t(-A)}A$. Hence $e^{-tA}\mathbf{y}(t) = const.$ on I and by $\mathbf{y}(0) = \mathbf{x}_0$, we obtain $\mathbf{y}(t) = e^{tA}\mathbf{x}_0$, $t \in I$. The proof is done. □

Remark 0.32 *As an application of the fundamental theorem, we can use it to prove that if $AB = BA$, then*

$$e^{A+B} = e^A e^B, \tag{27}$$

where A, B are two $n \times n$ real matrices. We consider the ODE

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = (A+B)\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{28}$$

The unique solution is given by $\mathbf{x}(t) = e^{t(A+B)}\mathbf{x}_0$. On the other hand, we also have

$$\begin{aligned} \frac{d}{dt}\left(e^{tA}e^{tB}\mathbf{x}_0\right) &= \left(\frac{d}{dt}e^{tA}\right)e^{tB}\mathbf{x}_0 + e^{tA}\left(\frac{d}{dt}e^{tB}\mathbf{x}_0\right) = \left(Ae^{tA}\right)e^{tB}\mathbf{x}_0 + e^{tA}\left(Be^{tB}\mathbf{x}_0\right) \\ &= Ae^{tA}e^{tB}\mathbf{x}_0 + Be^{tA}e^{tB}\mathbf{x}_0 = (A+B)e^{tA}e^{tB}\mathbf{x}_0, \end{aligned}$$

where we have used the identity $e^{tA}B = Be^{tA}$ (this is much easier to check than (27)). Thus $e^{tA}e^{tB}\mathbf{x}_0$ is also a solution of (28) (note that $e^{tA}e^{tB}\mathbf{x}_0\big|_{t=0} = \mathbf{x}_0$) and uniqueness implies

$$e^{tA}e^{tB}\mathbf{x}_0 = e^{t(A+B)}\mathbf{x}_0$$

for all $t \in (-\infty, \infty)$ and all $\mathbf{x}_0 \in \mathbb{R}^n$. In particular, we have

$$e^{tA}e^{tB} = e^{t(A+B)}, \quad \forall\, t \in (-\infty, \infty).$$

Letting $t = 1$, we have proved (27).

Exercise 0.33 *As an interesting application of Lemma 0.29, we can do the following:*

1. *Let $A,\ B$ be two $n \times n$ real matrices and we have*
$$e^{tA} = e^{tB}, \quad \forall\, t \in (-\infty, \infty)\,.$$
Prove or disprove that $A = B$.

2. *Let $A,\ B$ be two $n \times n$ real matrices and we have*
$$e^{A} = e^{B}.$$
Prove or disprove that $A = B$.

<u>**Solution**</u>:

(1). The answer is YES, i.e. $A = B$. To see this, apply d/dt to both sides and get
$$Ae^{tA} = Be^{tB}, \quad \forall\, t \in (-\infty, \infty)\,.$$
Letting $t = 0$, we get $A = B$.

(2). The answer is NO. It is possible to have $A \neq B$, but we still get $e^A = e^B$. As a simple example, choose
$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad \alpha = 0, \quad \beta = 2\pi.$$
Then
$$e^A = I, \quad e^B = \begin{pmatrix} e^{\alpha}\cos\beta & -e^{\alpha}\sin\beta \\ e^{\alpha}\sin\beta & e^{\alpha}\cos\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$
Similarly, we also have
$$e^B = I \quad \text{for any} \quad B = \begin{pmatrix} 0 & -2k\pi \\ 2k\pi & 0 \end{pmatrix}, \quad k \in \mathbb{N}.$$

□

0.2 2×2 linear system with constant coefficients.

The key point in the fundamental theorem is to compute the matrix e^{tA}. This is not easy when A is **not** diagonalizable. However, if A is a 2×2 matrix, then e^{tA} is not difficult to compute. We first have:

Lemma 0.34 *If A is a 2×2 real matrix, then there is an invertible real matrix P such that $P^{-1}AP$ has one of the forms*
$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{29}$$
for some real numbers $\lambda,\ \mu,\ \alpha,\ \beta$.

Remark 0.35 *We call (29) the **Jordan canonical forms** of A.*

Proof. Let $\lambda_1,\ \lambda_2 \in \mathbb{C}$ be the two eigenvalues of A. If $\lambda_1,\ \lambda_2$ are real and distinct, we have the first form. If $\lambda_1,\ \lambda_2$ are real and equal (call them λ), then there exists a nonzero vector $v_1 \in \mathbb{R}^2$ such that $Av_1 = \lambda v_1$. Let W be the subspace of $\mathbb{R}^2$ given by

$$W = \left\{ v \in \mathbb{R}^2 : Av = \lambda v \right\} = \ker\left(A - \lambda I\right).$$

If $W = \mathbb{R}^2$, then $A = \lambda I$ and we are in the first case again. Hence we assume that $\dim W = 1$ and choose **any nonzero vector** $w \in \mathbb{R}^2$ which is **independent** to v. Then we have

$$Aw = \alpha v_1 + \beta w \quad \text{for some number} \quad \alpha \neq 0,\ \beta.$$

Note that if $\alpha = 0$, then we have $Aw = \beta w,\ w \neq 0$, and so β is an eigenvalue (which must be the same as λ) and then we have two independent eigenvectors $v_1,\ w$ of λ, a contradiction. Hence $\alpha \neq 0$ and the two equation

$$\begin{cases} Av_1 = \lambda v_1 \\ Aw = \alpha v + \beta w \end{cases}$$

can be expressed as

$$A\left(v_1, w\right) = \left(v_1, w\right) \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq 0,$$

which is the same as

$$\left(v_1, w\right)^{-1} A\left(v_1, w\right) = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq 0. \tag{30}$$

Since the matrix in (30) is upper triangular, the number β must be an eigenvalue and is equal to λ. As a conclusion, we have

$$\begin{cases} Av_1 = \lambda v_1, \\ Aw = \alpha v_1 + \lambda w, \quad \alpha \neq 0 \end{cases} \tag{31}$$

and so

$$\begin{cases} Av_1 = \lambda v_1, \\ Av_2 = v_1 + \lambda v_2, \quad \text{where} \quad v_2 = \frac{1}{\alpha} w. \end{cases} \tag{32}$$

This gives the second case if we choose $P = \left(v_1, v_2\right)$.

Remark 0.36 *Another argument to derive (31): we already have $Av_1 = \lambda v_1$. Since $A \neq \lambda I$, there is a vector $w \neq 0$, independent to v_1, such that $Aw - \lambda w \neq 0$. Let $\sigma = Aw - \lambda w \neq 0$. By the* ***Cayley-Hamilton Theorem*** *in Linear Algebra, we know that*

$$\left(A - \lambda I\right)\sigma = \left(A - \lambda I\right)\left(A - \lambda I\right) w = \left(A - \lambda I\right)^2 w = 0.$$

Hence the nonzero vector σ must lie in the eigenspace of the eigenvalue λ and so $\sigma = \alpha v_1$ for some $\alpha \in \mathbb{R},\ \alpha \neq 0$. Thus we have

$$\begin{cases} Av_1 = \lambda v_1, \\ Aw = \alpha v_1 + \lambda w, \quad \alpha \neq 0, \end{cases}$$

which is the same as (31).

If $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \beta > 0$, then let $v_1 = u + iw,\ v_2 = u - iw,\ w \neq 0$, be corresponding **complex eigenvectors** of $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ respectively. We have $u,\ w \in \mathbb{R}^2$ and by

$$Av_1 = Au + iAw = \left(\alpha + i\beta\right)\left(u + iw\right) = \left(\alpha u - \beta w\right) + i\left(\alpha w + \beta u\right)$$

we have

$$Au = \alpha u - \beta w, \quad Aw = \beta u + \alpha w, \tag{33}$$

which also implies that $u,\ w$ are *linearly independent* in $\mathbb{R}^2$ (why? we first see that $u \neq 0$; then if u is a multiple of $w,\ A$ will have a **real** eigenvalue, a contradiction).

Now choose $P = (w, u)$ (note that here we change the order of $u,\ w$) and the above implies

$$A\,(w,u) = (w,u)\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

which gives the third case. Note that here we have changed the order of u and w. □

Remark 0.37 *(**Important.**) If we do not change order of u and w, we get*

$$P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

where now $P = (u, w)$ and $u + iw$ is the complex eigenvector of $\alpha + i\beta$. The reason that we prefer the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

instead of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

is that the we have the correspondence

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Longleftrightarrow (\alpha + i\beta)\,(x + iy)\,. \tag{34}$$

Remark 0.38 *(**Interesting.**) Let $\{u, w\}$ be a basis of $\mathbb{R}^2$. If we have*

$$\begin{cases} Au = \alpha u - \beta w \\ Aw = \beta u + \alpha w, \end{cases} \tag{35}$$

which is same as $A\,(u + iw) = (\alpha + i\beta)\,(u + iw)$ or

$$(w,u)^{-1}\,A\,(w,u) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

then we have (see Lemma 0.41 below)

$$(w,u)^{-1}\,e^A\,(w,u) = \begin{pmatrix} e^\alpha \cos\beta & -e^\alpha \sin\beta \\ e^\alpha \sin\beta & e^\alpha \cos\beta \end{pmatrix}.$$

Hence we get

$$e^A\,(w,u) = (w,u)\begin{pmatrix} e^\alpha \cos\beta & -e^\alpha \sin\beta \\ e^\alpha \sin\beta & e^\alpha \cos\beta \end{pmatrix},$$

i.e. we can conclude the following identities:

$$\begin{cases} e^A u = (e^\alpha \cos\beta)\,u - (e^\alpha \sin\beta)\,w \\ e^A w = (e^\alpha \sin\beta)\,u + (e^\alpha \cos\beta)\,w. \end{cases} \tag{36}$$

Example 0.39 *Reduce the matrix*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

to canonical form.

Solution: The matrix has two repeated eigenvalue $\lambda = 2$. Solve

$$\begin{cases} x - y = 2x \\ x + 3y = 2y \end{cases}$$

to get one eigenvector $v_1 = (1, -1)$. Choose $w = (1, 0)$ and get (note that one can choose any nonzero vector $w \in \mathbb{R}^2$ which is **independent** to v). Then we have

$$Aw = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha v_1 + 2w, \quad \alpha = -1,$$

where we note that the coefficient in front of w is 2, which is an eigenvalue (this must be the case as claimed in our proof).

According to the proof, if we choose $v_2 = \frac{1}{\alpha} w = -w = (-1, 0)$, we will have

$$Av_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = v_1 + 2v_2.$$

Hence

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$$

and

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

is the canonical form. □

Example 0.40 *Reduce the matrix*

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$$

to canonical form.

Solution: The matrix has two complex conjugate eigenvalues $\lambda = 2 \pm i \ (= \alpha \pm i\beta, \ \beta > 0)$. Solve

$$\begin{cases} 3x - 2y = (2 + i)\, x \\ x + y = (2 + i)\, y \end{cases}$$

to get $x = (1 + i)\, y$. Hence a complex eigenvector for $2 + i$ is (we take $y = 1$)

$$v = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = u + iw.$$

If we let

$$P = (w, u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \end{aligned}$$

which is a canonical form. □

Lemma 0.41 *If*

$$B=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix},$$

where $\lambda,\ \mu,\ \alpha,\ \beta$ are real numbers, then we have

$$e^{tB}=\begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{\mu t}\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}e^{\alpha t}\cos\beta t & -e^{\alpha t}\sin\beta t\\ e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t\end{pmatrix}$$

for all $t\in(-\infty,\infty)$.

Remark 0.42 *If*

$$B=\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}$$

then

$$e^{B}=e^{A+C}=e^{A}e^{C}=\begin{pmatrix}e^{a} & 0\\ 0 & e^{a}\end{pmatrix}\left(I+C+\frac{C^2}{2!}+\cdots\right),\quad A=\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix},\quad C=\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}e^{a} & 0\\ 0 & e^{a}\end{pmatrix}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix}e^{a} & 0\\ 0 & e^{a}\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}=\begin{pmatrix}e^{a} & be^{a}\\ 0 & e^{a}\end{pmatrix}.$$

Proof. The first case is trivial. For the second case, we have

$$tB=(\lambda t)\,I+C,\qquad C=\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix},\quad \text{where } (\lambda t)\,I \text{ and } C \text{ commute.}$$

Hence

$$e^{tB}=e^{(\lambda t)I}e^{C}=\begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{\lambda t}\end{pmatrix}\left(I+C+\frac{C^2}{2!}+\cdots\right)=\begin{pmatrix}e^{\lambda t} & 0\\ 0 & e^{\lambda t}\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}=\begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}.$$

For the third case, we have

$$tB=(\alpha t)\,I+C,\qquad C=\begin{pmatrix}0 & -\beta t\\ \beta t & 0\end{pmatrix}$$

and

$$e^{tB}=\begin{pmatrix}e^{\alpha t} & 0\\ 0 & e^{\alpha t}\end{pmatrix}\left(I+C+\frac{C^2}{2!}+\frac{C^3}{3!}+\cdots\right),$$

where

$$\begin{aligned}
&I+C+\frac{C^2}{2!}+\frac{C^3}{3!}+\cdots\\
&=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & -\beta t\\ \beta t & 0\end{pmatrix}+\frac{1}{2!}\begin{pmatrix}-(\beta t)^2 & 0\\ 0 & -(\beta t)^2\end{pmatrix}\\
&+\frac{1}{3!}\begin{pmatrix}0 & (\beta t)^3\\ -(\beta t)^3 & 0\end{pmatrix}+\frac{1}{4!}\begin{pmatrix}(\beta t)^4 & 0\\ 0 & (\beta t)^4\end{pmatrix}+\frac{1}{5!}\begin{pmatrix}0 & -(\beta t)^5\\ (\beta t)^5 & 0\end{pmatrix}+\cdots\\
&=\begin{pmatrix}1-\frac{(\beta t)^2}{2!}+\frac{(\beta t)^4}{4!}+\cdots & -(\beta t)+\frac{(\beta t)^3}{3!}-\frac{(\beta t)^5}{5!}+\cdots\\ (\beta t)-\frac{(\beta t)^3}{3!}+\frac{(\beta t)^5}{5!}+\cdots & 1-\frac{(\beta t)^2}{2!}+\frac{(\beta t)^4}{4!}+\cdots\end{pmatrix}=\begin{pmatrix}\cos(\beta t) & -\sin(\beta t)\\ \sin(\beta t) & \cos(\beta t)\end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{pmatrix}e^{\alpha t} & 0\\ 0 & e^{\alpha t}\end{pmatrix}\begin{pmatrix}\cos(\beta t) & -\sin(\beta t)\\ \sin(\beta t) & \cos(\beta t)\end{pmatrix}=\begin{pmatrix}e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t)\\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t)\end{pmatrix}.$$

The proof is done. □

Corollary 0.43 *For any 2×2 real matrix A we have*

$$\det e^{A} = e^{TrA}, \tag{37}$$

where TrA denotes the trace of A. In particular, we have

$$\det e^{tA} = e^{t(TrA)} \quad \textit{for all} \quad t \in (-\infty, \infty). \tag{38}$$

Remark 0.44 *The above corollary is actually valid for any $n \times n$ real matrix A. We shall prove this later on.*

Proof. Choose P such that $P^{-1}AP = B$ has one of the forms in (29). Then $e^{A} = Pe^{B}P^{-1}$, where

$$e^{B} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\alpha}\cos\beta & -e^{\alpha}\sin\beta \\ e^{\alpha}\sin\beta & e^{\alpha}\cos\beta \end{pmatrix}$$

and we also know that $TrA = TrB,\ \det A = \det B,\ Tre^{A} = Tre^{B},\ \det e^{A} = \det e^{B}$. Now, in any case, we have $\det e^{B} = e^{TrB}$, and so

$$\det e^{A} = e^{TrA}.$$

The proof is done. □

Example 0.45 *Assume that A is a 2×2 matrix. Is it possible to have*

$$e^{tA} = \begin{pmatrix} 0 & e^{t} \\ e^{2t} & 0 \end{pmatrix}$$

for some $t \in \mathbb{R}$? Give your reasons.

<u>**Solution**</u>:

By the identity $\det e^{tA} = e^{(Tr\ A)t}$ we must have $\det e^{tA} > 0$ for any matrix A and any $t \in \mathbb{R}$. But now

$$\det e^{tA} = -e^{3t} < 0.$$

Hence it is impossible. □

By the previous lemmas, we have:

Corollary 0.46 *Consider the 2×2 linear system*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^2, \end{cases} \tag{39}$$

where A has 2 repeated eigenvalue *λ and $A \neq \lambda I$. Then the solution (in matrix form) is given by*

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 = P\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}P^{-1}\mathbf{x}_0 \tag{40}$$

where P is any 2×2 invertible matrix satisfying

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In particular, if $P = (v_1, v_2)$, then $Av_1 = \lambda v_1,\ Av_2 = v_1 + \lambda v_2$, and $\mathbf{x}(t)$ can also be written (in vector form) as

$$\mathbf{x}(t) = c_1 e^{\lambda t}v_1 + c_2\left(te^{\lambda t}v_1 + e^{\lambda t}v_2\right), \tag{41}$$

where $c_1,\ c_2$ solves

$$c_1v_1 + c_2v_2 = \mathbf{x}_0 \quad \textit{(this is same as } P^{-1}\mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Remark 0.47 *Note that the coefficient of $te^{\lambda t}$ is the eigenvector v_1.*

Remark 0.48 *In the above corollary we have*

$$(A - \lambda I)\, v_1 = 0 \quad \text{and} \quad (A - \lambda I)\, v_2 = v_1. \tag{42}$$

In Linear Algebra book, the vector v_2 is also called a ***generalized eigenvector****.*

We also have:

Corollary 0.49 *Consider the 2×2 linear system*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^2, \end{cases} \tag{43}$$

where A has 2 complex conjugate eigenvalues $\alpha + i\beta,\ \alpha - i\beta,\ \beta > 0$. Then the solution (in matrix form) is given by

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 = P \begin{pmatrix} e^{\alpha t}\cos(\beta t) & -e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix} P^{-1}\mathbf{x}_0 \tag{44}$$

where P is any 2×2 invertible matrix satisfying

$$P^{-1}AP = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

In particular, if $P = (v_1, v_2)$ (note that now the eigenvector of $\lambda = \alpha + i\beta$ is $v_2 + iv_1$), then

$$Av_1 = \alpha v_1 + \beta v_2, \qquad Av_2 = -\beta v_1 + \alpha v_2, \tag{45}$$

and $\mathbf{x}(t)$ can also be written (in vector form) as

$$\mathbf{x}(t) = c_1\left[e^{\alpha t}\cos(\beta t)\cdot v_1 + e^{\alpha t}\sin(\beta t)\cdot v_2\right] + c_2\left[-e^{\alpha t}\sin(\beta t)\cdot v_1 + e^{\alpha t}\cos(\beta t)\cdot v_2\right], \tag{46}$$

where $c_1,\ c_2$ solves

$$c_1 v_1 + c_2 v_2 = \mathbf{x}_0.$$

Remark 0.50 *(**another method**) This is to use **complex solutions** and take its **real part** to get real solutions. Assume A has 2 complex conjugate eigenvalues $\alpha + i\beta,\ \alpha - i\beta,\ \beta > 0$, with corresponding complex eigenvectors $v = u + iw$ and $\bar{v} = u - iw$, where $u,\ w \in \mathbb{R}^2$. Then the general complex solution of $d\mathbf{x}/dt = A\mathbf{x}$ is*

$$\mathbf{x}(t) = c_1 e^{(\alpha + i\beta)t}(u + iw) + c_2 e^{(\alpha - i\beta)t}(u - iw),$$

*where $c_1 = a_1 + ib_1,\ c_2 = a_2 + ib_2$ are two arbitrary **complex** constants. Note that*

$$\begin{aligned}
& c_1 e^{(\alpha + i\beta)t}(u + iw) + c_2 e^{(\alpha - i\beta)t}(u - iw) \\
&= (a_1 + ib_1)\, e^{(\alpha + i\beta)t}(u + iw) + (a_2 + ib_2)\, e^{(\alpha - i\beta)t}(u - iw) \\
&= e^{\alpha t}(a_1 + ib_1)\underbrace{(\cos\beta t + i\sin\beta t)(u + iw)} + e^{\alpha t}(a_2 + ib_2)\underbrace{(\cos\beta t - i\sin\beta t)(u - iw)} \\
&= \begin{cases} e^{\alpha t}(a_1 + ib_1)\{[(\cos\beta t)\, u - (\sin\beta t)\, w] + i\,[(\sin\beta t)\, u + (\cos\beta t)\, w]\} \\ + e^{\alpha t}(a_2 + ib_2)\{[(\cos\beta t)\, u - (\sin\beta t)\, w] - i\,[(\sin\beta t)\, u + (\cos\beta t)\, w]\} \end{cases}
\end{aligned}$$

*The **real part** of the above complex solution is given by*

$$\begin{aligned}
&\begin{cases} e^{\alpha t}\left\{a_1\left[(\cos\beta t)\,u-(\sin\beta t)\,w\right]-b_1\left[(\sin\beta t)\,u+(\cos\beta t)\,w\right]\right\} \\ +e^{\alpha t}\left\{a_2\left[(\cos\beta t)\,u-(\sin\beta t)\,w\right]+b_2\left[(\sin\beta t)\,u+(\cos\beta t)\,w\right]\right\} \end{cases} \\
&=(a_1+a_2)\,e^{\alpha t}\left[(\cos\beta t)\,u-(\sin\beta t)\,w\right]+(b_2-b_1)\,e^{\alpha t}\left[(\sin\beta t)\,u+(\cos\beta t)\,w\right]
\end{aligned}$$

and since $a_1,\ a_2,\ b_1,\ b_2$ are all arbitrary, we obtain the general real solution

$$c_1e^{\alpha t}\left[(\cos\beta t)\,u-(\sin\beta t)\,w\right]+c_2e^{\alpha t}\left[(\sin\beta t)\,u+(\cos\beta t)\,w\right] \tag{47}$$

for arbitrary real constants $c_1,\ c_2$. Note that (47) is same as (46) if we replace u by v_2 and w by v_1.

Example 0.51 *(See Example 0.39 first.) Consider the linear system*

$$\begin{cases} x_1'(t)=x_1-x_2 \\ x_2'(t)=x_1+3x_2, \end{cases} \qquad A=\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{48}$$

We have $\lambda_1=\lambda_2=2$ and the canonical form

$$P^{-1}AP=\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where

$$P=(v_1,v_2)=\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \quad \textit{with} \quad Av_1=2v_1,\ Av_2=v_1+2v_2.$$

The general solution is given by (in vector form)

$$\begin{aligned}
\mathbf{x}(t)&=c_1e^{\lambda t}v_1+c_2\left(te^{\lambda t}v_1+e^{\lambda t}v_2\right) \\
&=c_1e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}+c_2\left(te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}+e^{2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right),
\end{aligned}$$

i.e.

$$x_1(t)=c_1e^{2t}+c_2(t-1)\,e^{2t}, \quad x_2(t)=-c_1e^{2t}-c_2te^{2t},$$

where $c_1,\ c_2$ are arbitrary constants.

Example 0.52 *(See Example 0.40 first.) Consider the linear system*

$$\begin{cases} x_1'(t)=3x_1-2x_2 \\ x_2'(t)=x_1+x_2, \end{cases} \qquad A=\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \tag{49}$$

We have $\lambda_1=2+i,\ \lambda_2=2-i$, and the canonical form

$$P^{-1}AP=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}=\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where

$$P=(v_1,v_2)=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{with} \quad Av_1=\alpha v_1+\beta v_2,\ Av_2=-\beta v_1+\alpha v_2.$$

The general solution is given by (in vector form)

$$\begin{aligned}
\mathbf{x}(t)&=c_1\left[e^{\alpha t}\cos(\beta t)\cdot v_1+e^{\alpha t}\sin(\beta t)\cdot v_2\right]+c_2\left[-e^{\alpha t}\sin(\beta t)\cdot v_1+e^{\alpha t}\cos(\beta t)\cdot v_2\right] \\
&=c_1\left[e^{2t}\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix}+e^{2t}\sin t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]+c_2\left[-e^{2t}\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}+e^{2t}\cos t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right],
\end{aligned}$$

where $c_1,\ c_2$ are arbitrary constants.

0.3 Some fact from linear algebra.

Lemma 0.53 *Let $A,\ B$ be two $n\times n$ real matrices with $B=(\mathbf{b}_1,\ ...,\ \mathbf{b}_n)$ where each $\mathbf{b}_i$ is a column vector. Then*

$$\det(A\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\det(\mathbf{b}_1,\ A\mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\cdots+\det(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ A\mathbf{b}_n)=TrA\cdot\det B, \quad (50)$$

where TrA denotes the trace of A.

Proof. Define the map $F:\mathbb{R}^n\times\cdots\times\mathbb{R}^n\to\mathbb{R}$ by

$$\begin{aligned}&F(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)\\&=\det(A\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\det(\mathbf{b}_1,\ A\mathbf{b}_2,\ ...,\ \mathbf{b}_n)+\cdots+\det(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ A\mathbf{b}_n).\end{aligned}$$

One can check that F is an **alternating multilinear map.** In particular, we have $F(\mathbf{b}_1,\ \mathbf{b}_2,\ ...,\ \mathbf{b}_n)=0$ if $\mathbf{b}_i=\mathbf{b}_j$ for some $i\neq j$.

In view of this, it suffices to check that (50) holds for the case $B=(\mathbf{e}_1,\ ...,\ \mathbf{e}_n)$, where $\{\mathbf{e}_1,\ ...,\ \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. But that is obvious. The proof is done. □

Lemma 0.54 *Let $A(t)$ be a time-dependent $n\times n$ real matrix which is* invertible *for all $t\in I$ (some interval). Then we have the identity*

$$\frac{d}{dt}\det A(t)=Tr\left(A^{-1}(t)\frac{dA}{dt}\right)\cdot\det A(t),\qquad\forall\ t\in I, \quad (51)$$

where we note that $Tr\left(A^{-1}(t)\frac{dA}{dt}\right)=Tr\left(\frac{dA}{dt}A^{-1}(t)\right)$.

Proof. This is a consequence of the previous lemma. Write $A(t)=(\mathbf{a}_1(t),\ \mathbf{a}_2(t),\ ...,\ \mathbf{a}_n(t))$, where $\mathbf{a}_i(t)$ are column vectors. Then

$$\begin{aligned}&\frac{d}{dt}\det A(t)\\&=\det(\mathbf{a}_1'(t),\ \mathbf{a}_2(t),\ ...,\ \mathbf{a}_n(t))+\det(\mathbf{a}_1(t),\ \mathbf{a}_2'(t),\ ...,\ \mathbf{a}_n(t))+\cdots+\det(\mathbf{a}_1(t),\ \mathbf{a}_2(t),\ ...,\ \mathbf{a}_n'(t))\end{aligned}$$

and we note that

$$A'(t)=(\mathbf{a}_1'(t),\ \mathbf{a}_2'(t),\ ...,\ \mathbf{a}_n'(t))$$

and if we let $P(t)=\frac{dA}{dt}A^{-1}(t)$, then

$$P(t)\mathbf{a}_1(t)=\left(\frac{dA}{dt}A^{-1}(t)\right)\mathbf{a}_1(t)=\frac{dA}{dt}\left(A^{-1}(t)\mathbf{a}_1(t)\right)=\frac{dA}{dt}(1,0,...,0)^T=\mathbf{a}_1'(t)$$

and similarly

$$P(t)\mathbf{a}_2(t)=\left(\frac{dA}{dt}A^{-1}(t)\right)\mathbf{a}_2(t)=\frac{dA}{dt}\left(A^{-1}(t)\mathbf{a}_2(t)\right)=\frac{dA}{dt}(0,1,0,...,0)^T=\mathbf{a}_2'(t),\quad etc.$$

Hence by Lemma 0.53 we have

$$\begin{aligned}&\frac{d}{dt}\det A(t)\\&=\begin{cases}\det(P(t)\mathbf{a}_1(t),\ \mathbf{a}_2(t),\ ...,\ \mathbf{a}_n(t))+\det(\mathbf{a}_1(t),\ P(t)\mathbf{a}_2(t),\ ...,\ \mathbf{a}_n(t))\\+\cdots+\det(\mathbf{a}_1(t),\ \mathbf{a}_2(t),\ ...,\ P(t)\mathbf{a}_n(t))\end{cases}\\&=Tr\ (P(t))\cdot\det A(t)=Tr\left(\frac{dA}{dt}A^{-1}(t)\right)\cdot\det A(t)=Tr\left(A^{-1}(t)\frac{dA}{dt}\right)\cdot\det A(t).\end{aligned}$$

The proof is done. □

Lemma 0.55 *Let A be any $n \times n$ real matrix, then*

$$\det e^{tA} = e^{t(TrA)} \quad \textit{for all} \quad t \in (-\infty, \infty). \tag{52}$$

and when $t = 1$, we get

$$\det e^{A} = e^{TrA}. \tag{53}$$

Remark 0.56 *Note that we always have*

$$\det e^{tA} > 0$$

for all $t \in (-\infty, \infty)$ and all real matrices A.

Proof. Compute

$$\frac{d}{dt} \det e^{tA} = \underline{Tr\left(B^{-1}(t) \frac{dB}{dt}\right)} \cdot \det e^{tA}, \quad B(t) = e^{tA}.$$

Since $B^{-1}(t) = e^{-tA}$ and $\frac{dB}{dt} = Ae^{tA} = e^{tA}A$, we have

$$\underline{Tr\left(B^{-1}(t) \frac{dB}{dt}\right)} = Tr\left(e^{-tA}e^{tA}A\right) = TrA.$$

Hence

$$\frac{d}{dt} \det e^{tA} = TrA \cdot \det e^{tA}, \quad \forall\, t \in (-\infty, \infty)$$

and so

$$\det e^{tA} = Ce^{(TrA)t}, \quad \forall\, t \in (-\infty, \infty)$$

for some constant C. Letting $t = 0$, we see that $C = 1$. The proof is done. $\square$

Corollary 0.57 *For any $n \times n$ real matrices $A,\ B$, we have the following:*

$$\begin{cases} (1).\ \left(e^{A}\right)^{-1} = e^{-A}. \\ (2).\ \left(e^{A}\right)^{T} = e^{A^{T}}. \\ (3).\ \det e^{A} = e^{TrA}. \\ (4).\ e^{A+B} = e^{A}e^{B} = e^{B}e^{A} = e^{B+A} \quad \textit{if} \quad AB = BA. \end{cases} \tag{54}$$

In general, there is no identity for $Tr\left(e^{A}\right)$. However, if A has n distinct real eigenvalues $\lambda_1,\ \ldots,\ \lambda_n$, then

$$\det e^{A} = e^{\lambda_1 + \cdots + \lambda_n} \quad \textit{and} \quad Tre^{A} = e^{\lambda_1} + \cdots + e^{\lambda_n}.$$

Remark 0.58 *We also have the following elementary fact: if $B = P^{-1}AP$, where $A,\ B,\ P$ are $n \times n$ real matrices, then*

$$\det B = \det A, \quad TrB = TrA, \quad e^{B} = Pe^{A}P^{-1}, \quad \det e^{B} = \det e^{A}, \quad Tre^{B} = Tre^{A}. \tag{55}$$

Proof. This is now clear. $\square$

Corollary 0.59 *If an $n \times n$ real matrix A satisfies $A^{T} = -A$ (i.e., A is **anti-symmetric**), then e^{A} is an **orthogonal** matrix.*

Remark 0.60 *If A is anti-symmetric, then all of its diagonal elements are zero. In particular, we have $TrA = 0$. We also have*

$$\det A = \det A^T = \det(-A) = (-1)^n \det A.$$

Hence if n is odd, we have $\det A = 0$.

Proof. Let $M = e^A$. Then, by definition, M is an orthogonal matrix if and only if it satisfies $M^T = M^{-1}$. We now have

$$M^T = \left(e^A\right)^T = e^{A^T} = e^{-A} = M^{-1}.$$

The proof is done. □

Lemma 0.61 *Assume that A is an $n \times n$ anti-symmetric real matrix. Then for any two solutions $\mathbf{x}^{(1)}(t)\,,\ \mathbf{x}^{(2)}(t) \in \mathbb{R}^n$ to the linear system of equations*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

their inner product $\left\langle \mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t) \right\rangle$ is independent of *time.*

Proof. By

$$\frac{d}{dt}\left\langle \mathbf{x}^{(1)}(t)\,,\ \mathbf{x}^{(2)}(t) \right\rangle = \left\langle A\mathbf{x}^{(1)}(t)\,,\ \mathbf{x}^{(2)}(t) \right\rangle + \left\langle \mathbf{x}^{(1)}(t)\,,\ A\mathbf{x}^{(2)}(t) \right\rangle = \left\langle \left(A + A^T\right)\mathbf{x}^{(1)}(t)\,,\ \mathbf{x}^{(2)}(t) \right\rangle = 0$$

the conclusion is proved. □

Remark 0.62 *Another proof is: Since e^{tA} is an* ***orthogonal*** *matrix, we have*

$$\left\langle \mathbf{x}^{(1)}(t)\,,\ \mathbf{x}^{(2)}(t) \right\rangle = \left\langle e^{tA}\mathbf{x}^{(1)}(0)\,,\ e^{tA}\mathbf{x}^{(2)}(0) \right\rangle = \left\langle \left(e^{tA}\right)^T e^{tA}\mathbf{x}^{(1)}(0)\,,\ \mathbf{x}^{(2)}(0) \right\rangle = \left\langle \mathbf{x}^{(1)}(0)\,,\ \mathbf{x}^{(2)}(0) \right\rangle$$

for all $t \in \mathbb{R}$. In particular, we see that if A is an $n \times n$ anti-symmetric real matrix, the orthogonal linear transformation $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ preserves length and volume *for each fixed time $t \in \mathbb{R}$. We call the map $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n,\ t \in (-\infty, \infty)$, the* ***flow*** *generated by the ODE $d\mathbf{x}/dt = A\mathbf{x}$.*

Lemma 0.63 *Assume that A is a real $n \times n$ anti-symmetric matrix. Then its eigenvalues are either* 0 *or* ***pure imaginary.***

Remark 0.64 *Compare with the well-known fact: if A is a real $n \times n$* ***symmetric*** *matrix, then all of its eigenvalues are* ***real.***

Proof. Let $\lambda \in \mathbb{R}$ be a real eigenvalue. Then there exists some nonzero $v \in \mathbb{R}^n$ such that $Av = \lambda v$. Hence

$$\lambda |v|^2 = \langle \lambda v,\ v \rangle = \langle Av,\ v \rangle = \left\langle v,\ A^T v \right\rangle = \langle v,\ -Av \rangle = \langle v,\ -\lambda v \rangle = -\langle v,\ \lambda v \rangle = -\lambda |v|^2,$$

which implies that $\lambda = 0$.

On the other hand, if λ is a complex eigenvalue, then there exists some nonzero *complex* eigenvector $v \in \mathbb{C}^n$ such that $Av = \lambda v$. Using complex inner product $\langle , \rangle_{\mathbb{C}}$ we have

$$\langle Av,\ v \rangle_{\mathbb{C}} = \left\langle v,\ \overline{A^T} v \right\rangle_{\mathbb{C}}$$

and so (note that A is a real matrix)

$$\lambda |v|^2 = \langle \lambda v,\ v \rangle_{\mathbb{C}} = \langle Av,\ v \rangle_{\mathbb{C}} = \left\langle v,\ \overline{A^T} v \right\rangle_{\mathbb{C}} = \left\langle v,\ \overline{-A} v \right\rangle_{\mathbb{C}} = -\langle v,\ Av \rangle_{\mathbb{C}} = -\langle v,\ \lambda v \rangle_{\mathbb{C}} = -\bar{\lambda} |v|^2$$

and so $\lambda + \bar{\lambda} = 0$. Thus λ is **pure imaginary**. □

0.4 Nonhomogeneous 2×2 linear system.

Let A be an $n\times n$ real matrix. We now consider the equation

$$\begin{cases} \dfrac{d\mathbf{x}}{dt}=A\mathbf{x}+\mathbf{b}\left(t\right), \quad t\in I, \quad 0\in I \\ \mathbf{x}\left(0\right)=\mathbf{x}_0\in\mathbb{R}^n \end{cases} \tag{56}$$

where $\mathbf{b}\left(t\right)\in\mathbb{R}^n$ is a continuous function defined on some interval I with $0\in I$.

Theorem 0.65 *The solution to (56) is unique and is defined on I, given by the following "**general solution formula**":*

$$\mathbf{x}\left(t\right)=e^{tA}\mathbf{x}_0+e^{tA}\int_0^t e^{-sA}\mathbf{b}\left(s\right)ds, \quad t\in I. \tag{57}$$

Remark 0.66 *(57) is the same as the general solution formula in the one-dimensional case.* It is good only for theoretical purpose.

Proof. We have

$$\begin{aligned}\frac{d\mathbf{x}}{dt}&=Ae^{tA}\mathbf{x}_0+Ae^{tA}\int_0^t e^{-sA}\mathbf{b}\left(s\right)ds+e^{tA}e^{-tA}\mathbf{b}\left(t\right)\\&=A\left(e^{tA}\mathbf{x}_0+e^{tA}\int_0^t e^{-sA}\mathbf{b}\left(s\right)ds\right)+\mathbf{b}\left(t\right)=A\mathbf{x}+\mathbf{b}\left(t\right), \quad t\in I.\end{aligned}$$

As for uniqueness, if we have two solutions to (56) on I, their difference $\mathbf{w}\left(t\right)$ will satisfy

$$\frac{d\mathbf{w}}{dt}=A\mathbf{w}, \quad \mathbf{w}\left(0\right)=0.$$

Hence, by uniqueness, we must have $\mathbf{w}\left(t\right)\equiv 0$. The proof is done. $\square$

In practice, we will prefer to use "**diagonalization method** (**decoupled method**)" if A has 2 distinct real eigenvalues or 2 repeated real eigenvalues. However, if A has 2 complex conjugate eigenvalues, the method is slightly different.

Example 0.67 *(**2 different real eigenvalues.**) Find the general solution of the equation*

$$\frac{d\mathbf{x}}{dt}=\begin{pmatrix}3 & -1\\ 4 & -2\end{pmatrix}\mathbf{x}+\begin{pmatrix}2e^{-t}\\ 3t\end{pmatrix}, \quad t\in\left(-\infty,\infty\right).$$

<u>**Solution**</u>:

We have

$$A=\begin{pmatrix}3 & -1\\ 4 & -2\end{pmatrix}$$

with eigenvalues $\lambda_1=2,\ \lambda_2=-1$ and corresponding eigenvectors $v_1=\left(1,1\right),\ v_2=\left(1,4\right)$. Thus

$$P=\begin{pmatrix}1 & 1\\ 1 & 4\end{pmatrix}, \quad P^{-1}=\frac{1}{3}\begin{pmatrix}4 & -1\\ -1 & 1\end{pmatrix}, \quad P^{-1}AP=\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}$$

and if we let

$$\mathbf{x}=P\mathbf{y}$$

we would have

$$\frac{d\mathbf{x}}{dt}=P\frac{d\mathbf{y}}{dt}=A\mathbf{x}+\begin{pmatrix}2e^{-t}\\ 3t\end{pmatrix}=AP\mathbf{y}+\begin{pmatrix}2e^{-t}\\ 3t\end{pmatrix}$$

and so

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y} + P^{-1}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} + P^{-1}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix},$$

which gives (the system becomes **decoupled**)

$$\begin{cases} \dfrac{dy_1}{dt} = 2y_1 + \frac{1}{3}\left(8e^{-t} - 3t\right) \\ \dfrac{dy_2}{dt} = -y_2 + \frac{1}{3}\left(-2e^{-t} + 3t\right). \end{cases}$$

The solution of the above is

$$\begin{cases} y_1(t) = C_1 e^{2t} - \frac{8}{9}e^{-t} + \frac{1}{2}t + \frac{1}{4} \\ y_2(t) = C_2 e^{-t} - \frac{2}{3}te^{-t} + t - 1. \end{cases}$$

Finally we get the general solution

$$\mathbf{x}(t) = P\mathbf{y}(t) = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} C_1 e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} \\ C_2 e^{-t} + te^{-t} + \frac{3}{2}t - \frac{3}{2} \end{pmatrix}.$$

□

Example 0.68 *(**2 different real eigenvalues.**) Find the general solution of the equation*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \quad t \in (-\infty, \infty).$$

Solution:

The two eigenvalues of the coefficients matrix A are $\lambda_1 = -3$, $\lambda_2 = -1$, with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we let

$$\mathbf{x} = P\mathbf{y}$$

we would have

$$\frac{d\mathbf{x}}{dt} = P\frac{d\mathbf{y}}{dt} = A\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = AP\mathbf{y} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

and so

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y} + P^{-1}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix},$$

which gives (the system becomes **decoupled**)

$$\begin{cases} \dfrac{dy_1}{dt} = -3y_1 + \frac{1}{2}\left(2e^{-t} - 3t\right) \\ \dfrac{dy_2}{dt} = -y_2 + \frac{1}{2}\left(2e^{-t} + 3t\right) \end{cases}$$

we get

$$\begin{cases} y_1(t) = C_1 e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6}, \\ y_2(t) = C_2 e^{-t} + te^{-t} + \frac{3}{2}t - \frac{3}{2}. \end{cases}$$

Finally we get the general solution

$$\mathbf{x}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} \\ C_2 e^{-t} + te^{-t} + \frac{3}{2}t - \frac{3}{2} \end{pmatrix}.$$

□

Example 0.69 *(**2 repeated real eigenvalues.**) Find the general solution of the equation*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \quad t \in (-\infty, \infty).$$

Solution:

See Example 0.39also. We have $\lambda_1 = \lambda_2 = 2$ with $v_1 = (1, -1)$, $v_2 = (-1, 0)$, where

$$Av_1 = 2v_1, \quad Av_2 = v_1 + 2v_2.$$

Hence

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}. \tag{58}$$

Let $\mathbf{x} = P\mathbf{y}$ to get

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y} + P^{-1}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

and then (the system becomes **"semi-decoupled"**)

$$\begin{cases} \dfrac{dy_1}{dt} = 2y_1 + y_2 - 3t \\ \dfrac{dy_2}{dt} = 2y_2 - 2e^{-t} - 3t. \end{cases}$$

One can **solve the second equation first** and then plug into the first equation to solve it (one can always do so, as guaranteed by the canonical form (58)). Finally we have

$$\mathbf{x}(t) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

We leave the details to you. □

Example 0.70 *(**2 complex conjugate eigenvalues.**) Find the general solution of the equation*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \quad t \in (-\infty, \infty).$$

Solution:

See Example 0.40 also. The matrix has eigenvalues $2 \pm i$. A complex eigenvector for $2 + i$ is

$$v = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u + iw.$$

Now

$$P = (w, u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

and the system in terms of $\mathbf{y}(t)$ $(\mathbf{x} = P\mathbf{y})$ is given by

$$\begin{cases} \dfrac{dy_1}{dt} = 2y_1 - y_2 + 2e^{-t} - 3t \\ \dfrac{dy_2}{dt} = y_1 + 2y_2 + 3t. \end{cases}$$

Unfortunately this system is **not** decoupled (*however, after change of variables, it has a better symmetric form to work on*). There is one way to avoid the use of formula (57), but we still have to do a lot of computation. Rewrite the above as

$$\begin{cases} (D-2)\, y_1 + y_2 = 2e^{-t} - 3t \\ -y_1 + (D-2)\, y_2 = 3t, \end{cases} \tag{59}$$

where the operator D means $\frac{d}{dt}$, and if apply the operator $(D-2)$ to the second equation and add it to the first equation, we would get

$$(D-2)^2\, y_2 + y_2 = 2e^{-t} - 3t + (D-2)\, 3t = 2e^{-t} - 9t + 3,$$

i.e.,

$$y_2''(t) - 4y_2'(t) + 5y_2(t) = 2e^{-t} - 9t + 3.$$

From it we can solve $y_2(t)$ (use **undetermined coefficient method** or **variation of parameters method**) and plug it into the **second** equation of (59) to solve $y_1(t)$ (be careful: it will be too much trouble if we plug $y_2(t)$ into the first equation of (59) to solve $y_1(t)$). We leave the details to you ... □

0.5 3×3 linear system with constant coefficients.

The ODE to be solved now is the following 3×3 linear system with constant coefficients:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where } A \text{ is a } 3 \times 3 \text{ constant real matrix.} \tag{60}$$

By theory, we know that the solution is given by

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0, \quad t \in (-\infty, \infty), \tag{61}$$

where $\mathbf{x}_0$ is the initial condition.

In the following, we want to use the "**diagonalization method** (**decoupled method**)" to solve it. Denote the three eigenvalues of A by λ_1, λ_2 and λ_3. We have several cases to consider.

Before going on, we recall two important facts from Linear Algebra:

Lemma 0.71 *Let A be an $n \times n$ real matrix with characteristic polynomial*

$$P_n(\lambda) = \det(A - \lambda I), \quad \deg P_n(\lambda) = n.$$

If $\lambda = \lambda_0$ is a root of $P_n(\lambda) = 0$ with multiplicity m (i.e. λ_0 is a root which appears m times), $m \in \{1, 2, ..., n\}$, then we have

$$\dim \ker (A - \lambda_0 I) \le m, \tag{62}$$

where $\ker(A - \lambda_0 I) := \{v \in \mathbb{R}^n : (A - \lambda_0 I)\, v = 0\}$ *is the* eigenspace *of* λ_0.

Remark 0.72 *The above is also known as: "geometric multiplicity" $\le$ "algebraic multiplicity".*

Lemma 0.73 *(**Rank Theorem**.) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation (n, m can be any two positive integers). Then we have*

$$\dim \operatorname{Im} T + \dim \ker T = n. \tag{63}$$

To solve (60), we divide our discussions into several cases.

0.5.1 Case 1: $\lambda_1,\ \lambda_2,\ \lambda_3$ are real and distinct.

This is the easiest case. Let $v_1,\ v_2,\ v_3$ be the corresponding eigenvectors of $\lambda_1,\ \lambda_2,\ \lambda_3$. Then they are independent. If we let

$$P = (v_1, v_2, v_3) \quad \text{(each } v_i \text{ is a column vector)},$$

then P is invertible with

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Now let $\mathbf{x} = P\mathbf{y}$ (change of variables) to get

$$AP\mathbf{y} = A\mathbf{x} = \frac{d\mathbf{x}}{dt} = P\frac{d\mathbf{y}}{dt}$$

and obtain the equation for $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))$, which is

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\mathbf{y}.$$

Thus one can easily solve $\mathbf{y}(t)$ to get $\mathbf{y}(t) = \left(c_1e^{\lambda_1 t},\ c_2e^{\lambda_2 t},\ c_3e^{\lambda_3 t}\right)$. By the relation $\mathbf{x} = P\mathbf{y}$, one can get the general solution $\mathbf{x}(t)$ of (60), i.e.,

$$\mathbf{x}(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2 + c_3e^{\lambda_3 t}v_3, \tag{64}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants.

Remark 0.74 *We also have*

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 = PD(t)P^{-1}\mathbf{x}_0, \quad D(t) = diag\left(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\right), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{65}$$

and if we write $P^{-1}\mathbf{x}_0$ as $P^{-1}\mathbf{x}_0 = (c_1, c_2, c_3)$, we get the same solution as in (64).

0.5.2 Case 2: $\lambda_1 = \lambda,\ \lambda_2 = \lambda_3 = \sigma,\ \lambda \neq \sigma,\ \lambda,\ \sigma \in \mathbb{R}$.

For this case, we have two subcases: either $\dim\ker(A - \sigma I) = 2$ or $\dim\ker(A - \sigma I) = 1$.

Case 2A: $\dim\ker(A - \sigma I) = 2$. In this case we can find **two** linearly independent eigenvectors $v_2,\ v_3$ for the repeated eigenvalue σ. Let v_1 be the corresponding eigenvector of λ, then we can diagonalize A as (it is easy to see that $v_1,\ v_2,\ v_3$ are linearly independent in $\mathbb{R}^3$)

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \quad P = (v_1, v_2, v_3).$$

Then we are in the previous easy case.

Case 2B: $\dim\ker(A - \sigma I) = 1$. In this case we can find only **one** independent eigenvector for the repeated eigenvalue σ. In this case we **cannot** diagonalize the matrix A. However, we have the following:

Lemma 0.75 *Assume that we can find only* **one** *independent eigenvector for the repeated eigenvalue* σ*. Then there exist* three linearly independent vectors $v_1,\ v_2,\ v_3$ *(where* $v_1,\ v_2$ *are* ***eigenvectors*** *of* λ *and* σ *respectively, and* v_3 *is a* ***generalized eigenvector*** *of* σ*) such that*

$$A(v_1, v_2, v_3) = (v_1, v_2, v_3)\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix} \tag{66}$$

where (v_1, v_2, v_3) *is the* 3×3 *matrix with column vectors* $v_1,\ v_2,\ v_3$*.*

Proof. Let $v_1,\ v_2$ be two independent eigenvectors $v_1,\ v_2$ with $Av_1 = \lambda v_1,\ Av_2 = \sigma v_2,\ \lambda \neq \sigma$. Consider the map

$$A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3. \tag{67}$$

Let $R = \operatorname{Im}(A - \sigma I),\ K = \ker(A - \sigma I)$ (K is the *eigenspace* of σ), $\dim K = 1$. By the **Rank Theorem** in Linear Algebra (applied to the linear transformation $A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3$), we know that $\dim R = 2$.

We claim: $K \subset R$ (note that K is a *line* and R is a *plane*).

If $K \not\subset R$, then the operator $A - \sigma I : R \to R$ (now we restrict $A - \sigma I$ onto the subspace $R \subset \mathbb{R}^3$) has zero kernel and thus *1-1*. By **Rank Theorem** again, it is also *onto*. Hence for any $v \in R$ there exists some $w \in R$ such that $(A - \sigma I)w = v$, which gives

$$Av = A(A - \sigma I)w = (A - \sigma I)(Aw) \quad (\text{note that } R \text{ is the image of } A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3)$$

This says that $Av \in R$ also. Hence

$$A : R \to R\ (A \text{ is a linear map from } R \text{ to } R,\ \dim R = 2) \tag{68}$$

and on it we have two eigenvalues $\beta_1,\ \beta_2$ (regardless of what they are). Since we assume $K = \ker(A - \sigma I) \not\subset R$, both eigenvalues of A on R must be different from σ. This will force (note that A has two eigenvalues λ and σ only)

$$\beta_1 = \beta_2 = \lambda,$$

which contradicts the fact that the eigenvalues of A are $\lambda,\ \sigma,\ \sigma$. Hence $K \subset R$ and the claim is proved.

As $K \subset R$, we have $v_2 \in K \subset R$. Hence there exists some vector $v_3 \neq 0 \in \mathbb{R}^3$ such that

$$(A - \sigma I)v_3 = v_2 \in K \quad (\text{i.e. } Av_3 = v_2 + \sigma v_3) \tag{69}$$

We called v_3 a **generalized eigenvector** of σ corresponding to v_2. It is *independent* to v_2.

We then claim that $v_1,\ v_2,\ v_3$ are linearly independent. If not, then (we already know that $v_1,\ v_2$ are independent)

$$v_3 = \alpha v_1 + \beta v_2 \quad \text{for some } \alpha,\ \beta.$$

Applying $A - \sigma I$ onto it to get

$$v_2 = (A - \sigma I)v_3 = (A - \sigma I)(\alpha v_1 + \beta v_2) = \alpha(A - \sigma I)v_1 = \alpha(\lambda - \sigma)v_1,$$

a contradiction. Therefore we have (66) and the proof is done. □

Remark 0.76 *(See Remark 0.48 first.) In the* 2×2 *case, we have* $K = R$ *(now both have dimension* 1*) due to (42). Moreover, we have* $(A - \lambda I)^2 = 0$ *(now* λ *is the repeated eigenvalue). This can also be seen from its canonical form since*

$$(A - \lambda I)^2 = \left[p^{-1}(A - \lambda I)P\right]^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

However, this is not the case in $\mathbb{R}^3$*.*

Remark 0.77 *(**Summary.**) In conclusion, we need to solve v_1 (eigenvector of λ), v_2 (eigenvector of σ, $v_2 \in K$), v_3 (**generalized eigenvector** of σ corresponding to v_2) satisfying the system:*

$$Av_1 = \lambda v_1, \qquad Av_2 = \sigma v_2, \qquad Av_3 = v_2 + \sigma v_3, \tag{70}$$

where in the third equation of (70), we need to use the fact

$$K = \ker(A - \sigma I) \subset R = \operatorname{Im}(A - \sigma I), \quad K \text{ is a line and } R \text{ is a plane.} \tag{71}$$

Since $v_2 \in K \subset R$, the equation $Av_3 = v_2 + \sigma v_3$ must have a solution for v_3.

Let $P = (v_1, v_2, v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x} = P\mathbf{y}$) becomes the following **"semi-decoupled system"**:

$$\frac{d\mathbf{y}}{dt}(t) = \left(P^{-1}AP\right)\mathbf{y} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}\mathbf{y}, \quad \mathbf{y} = (y_1, y_2, y_3) \tag{72}$$

and the general solution to the ODE is given by

$$\begin{aligned} \mathbf{x}(t) &= P\mathbf{y}(t) = (v_1, v_2, v_3)\,\mathbf{y}(t) = (v_1, v_2, v_3)\begin{pmatrix} c_1 e^{\lambda t} \\ (c_2 + c_3 t)\, e^{\sigma t} \\ c_3 e^{\sigma t} \end{pmatrix} \\ &= c_1 e^{\lambda t} v_1 + \underbrace{(c_2 + c_3 t)}\, e^{\sigma t} v_2 + c_3 e^{\sigma t} v_3. \end{aligned} \tag{73}$$

Note that in the above v_1 and v_2 are eigenvalue vectors.

Example 0.78 *Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{74}$$

Solution:

The eigenvalues of the coefficient matrix are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$. To find the eigenvector for $\lambda = 2$, we solve

$$\begin{cases} x = 2x \\ -4x + y = 2y \\ 3x + 6y + 2z = 2z \end{cases}$$

and we obtain $v_1 = (0, 0, 1)$. To find the eigenvector for the repeated $\sigma = 1$, we solve

$$\begin{cases} x = x \\ -4x + y = y \\ 3x + 6y + 2z = z \end{cases}$$

and we obtain one eigenvector $v_2 = (0, 1, -6)$. As it is impossible to find another independent eigenvector, we have to find **generalized eigenvector**. We solve

$$\begin{cases} x = 0 + x \\ -4x + y = 1 + y \\ 3x + 6y + 2z = -6 + z \end{cases}$$

and obtain $x=-1/4,\ -3/4+6y+z=-6$. Hence a **generalized eigenvector** is $v_3=(-1/4,-1,3/4)$ (or other possible answers). We see that $v_1,\ v_2,\ v_3$ are linearly independent.

The general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \underbrace{(c_2 + c_3 t)}\, e^t \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + c_3 e^t \begin{pmatrix} -\frac{1}{4} \\ -1 \\ \frac{3}{4} \end{pmatrix}. \tag{75}$$

□

Remark 0.79 *Another method: Since the matrix in (74) is lower triangular, one can solve $x(t)$ first and then use it to solve $y(t)$, and then use $x(t)$ and $y(t)$ to solve $z(t)$.*

0.5.3 $\lambda_1=\lambda_2=\lambda_3=\lambda$ (the most difficult case).

Case 1: The eigenspace $\ker(A-\lambda I)$ has dimension 2.

Remark 0.80 *Unless $A=\lambda I$, otherwise the case $\dim\ker(A-\lambda I)=3$ cannot happen.*

Assume $\lambda_1=\lambda_2=\lambda_3=\lambda$ and $\dim\ker(A-\lambda I)=2$. This means that we can find **two** linearly independent eigenvectors of λ.

We claim the following:

Lemma 0.81 *Assume that $\dim\ker(A-\lambda I)=2$. Then there exist three linearly independent vectors $v_1,\ v_2,\ v_3$ (where $v_1,\ v_2$ are **eigenvectors**) such that*

$$A(v_1,v_2,v_3)=(v_1,v_2,v_3)\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \tag{76}$$

where (v_1,v_2,v_3) is the 3×3 matrix with column vectors $v_1,\ v_2,\ v_3$.

Proof. Consider the map $A-\lambda I:\mathbb{R}^3\to\mathbb{R}^3$. Let $R=\operatorname{Im}(A-\lambda I)$, $K=\ker(A-\lambda I)$, $\dim K=2$. By the **Rank Theorem**, we know $\dim R=1$. **We claim that** $R\subset K$ (note that now K is a *plane* and R is a *line*). To see this, choose a nonzero vector $v\in R$, then $(A-\lambda I)v\in R$ also (note that now $A-\lambda I:R\to R$ with $\dim R=1$). Since $\dim R=1$, we must have

$$(A-\lambda I)v=\mu v \quad \text{for some} \quad \mu\in\mathbb{R}.$$

If $\mu\neq 0$, then A has eigenvalue $\lambda+\mu$, a contradiction. Hence $(A-\lambda I)v=0$ for all $v\in R$ and this implies $R\subset K$.

Now we choose two linearly independent vectors $v_1,\ v_2$ in K with $v_1\notin R,\ v_2\in R$. Then there exists some nonzero vector v_3 such that

$$(A-\lambda I)v_3=v_2\in R. \tag{77}$$

Such vector $v_3\notin K$ and so it is independent to $v_1,\ v_2$. We now have identity (76). □

Remark 0.82 ***(Summary.)*** *In conclusion, we need to solve v_1 (eigenvector of λ, $v_1\in K$, $v_1\notin R$), v_2 (eigenvector of λ, $v_2\in R\subset K$), v_3 (**generalized eigenvector** of λ corresponding to v_2, $v_3\notin K$) satisfying the system:*

$$Av_1=\lambda v_1, \qquad Av_2=\lambda v_2, \qquad Av_3=v_2+\lambda v_3. \tag{78}$$

This is similar to (70). In the third equation of (78), we need to use the fact

$$\begin{cases} R=\operatorname{Im}(A-\lambda I)\subset K=\ker(A-\lambda I), & R \text{ is a line and } K \text{ is a plane} \\ v_1,\ v_2\in K, \quad v_1\notin R,\ v_2\in R, & v_3\in\mathbb{R}^3. \end{cases} \tag{79}$$

Since $v_2\in R$, the equation $Av_3=v_2+\lambda v_3$ must have a solution for v_3.

Remark 0.83 *Note that we have*

$$(A-\lambda I)^2 v = 0 \quad \textit{for all} \quad v \in \mathbb{R}^3. \tag{80}$$

That is:

$$\mathbb{R}^3 \overset{A-\lambda I}{\to} R\ (R \subset K) \overset{A-\lambda I}{\to} 0. \tag{81}$$

Let $P = (v_1, v_2, v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x} = P\mathbf{y}$) becomes the "**semi-decoupled system**":

$$\frac{d\mathbf{y}}{dt}(t) = \left(P^{-1}AP\right)\mathbf{y} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = (y_1, y_2, y_3) \tag{82}$$

and, similar to (73), the general solution to the ODE is given by

$$\begin{aligned} \mathbf{x}(t) &= (v_1, v_2, v_3)\,\mathbf{y}(t) = (v_1, v_2, v_3) \begin{pmatrix} c_1 e^{\lambda t} \\ (c_2 + c_3 t)\, e^{\lambda t} \\ c_3 e^{\lambda t} \end{pmatrix} \\ &= c_1 e^{\lambda t} v_1 + \underbrace{(c_2 + c_3 t)}\, e^{\lambda t} v_2 + c_3 e^{\lambda t} v_3. \end{aligned} \tag{83}$$

Note that in the above v_1 and v_2 are eigenvectors.

Example 0.84 *Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

<u>Solution:</u>

The characteristic polynomial of the coefficient matrix is

$$\begin{aligned} &\begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} \\ &= (5-\lambda)(-5-\lambda)(3-\lambda) - 48 - 48 + 8(5+\lambda) + 12(5-\lambda) + 24(3-\lambda) \\ &= \left(\lambda^2 - 2\lambda + 1\right)(4-\lambda) + 4 - 3\lambda = -(\lambda-1)^3. \end{aligned}$$

Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. To find the eigenvector for $\lambda = 1$, we solve

$$\begin{cases} 5x - 3y - 2z = x \\ 8x - 5y - 4z = y \\ -4x + 3y + 3z = z \end{cases}$$

and obtain $4x - 3y - 2z = 0$. Thus one can find **two** linearly independent eigenvectors $v_1,\ v_2$. The space $K = \ker(A - I)$ is given by the plane $4x - 3y - 2z = 0$.

The image of the matrix

$$A - I = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

is a line R given by $\{t(1,2,-1) : t \in (-\infty, \infty)\}$. We note that $R \subset K$.

According to the proof, we must choose two linearly independent vectors $v_1,\ v_2$ in K with $v_1 \notin R,\ v_2 \in R$. Thus we choose $v_1 = (3,4,0),\ v_2 = (1,2,-1)$. Finally, we solve $Av_3 = v_2 + v_3$ to get

$$\begin{cases} 5x - 3y - 2z = 1 + x \\ 8x - 5y - 4z = 2 + y \\ -4x + 3y + 3z = -1 + z \end{cases}$$

and get $4x - 3y - 2z = 1$. So we choose $v_3 = (0,1,-2)$. We see that $v_1,\ v_2,\ v_3$ are linearly independent.

The general solution is given by

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \underbrace{(c_2 + c_3 t)}\, e^t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}. \tag{84}$$

□

Remark 0.85 ***(important)*** *If we do* **not** *choose $v_2 \in R$, then the system $Av_3 = v_2 + v_3$ may* **not** *have a solution. For example, choose $v_2 = (3,4,0) \in K,\ v_2 \notin R$. Then we solve*

$$\begin{cases} 5x - 3y - 2z = 3 + x \\ 8x - 5y - 4z = 4 + y \\ -4x + 3y + 3z = 0 + z \end{cases}$$

and see that there is no solution at all (see the first equation and the third equation).

Case 2: The eigenspace $\ker(A - \lambda I)$ has dimension 1.

Assume $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $\dim\ker(A - \lambda I) = 1$. This means that we can find **only one** independent eigenvector of λ.

In this case Lemma 0.81 becomes the following:

Lemma 0.86 *Assume* $\dim\ker(A - \lambda I) = 1$. *Then there exist three linearly independent vectors $v_1,\ v_2,\ v_3$ (where v_1 is* ***eigenvector****) such that*

$$A(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \tag{85}$$

where (v_1, v_2, v_3) is the 3×3 matrix with column vectors $v_1,\ v_2,\ v_3$.

Proof. Consider the map $A - \lambda I : \mathbb{R}^3 \to \mathbb{R}^3$. Let $R = \mathrm{Im}(A - \lambda I),\ K = \ker(A - \lambda I),\ \dim R = 2,\ \dim K = 1$. We claim that $K \subset R$ (now K is a *line* and R is a *plane*). If not, then the operator $A - \lambda I : R \to R$ has zero kernel and thus *1-1*. By **Rank Theorem** again, it is also *onto*. Hence for any $v \in R$ there exists some $w \in R$ such that

$$(A - \lambda I)\, w = v,$$

which gives

$$Av = A(A - \lambda I)\, w = (A - \lambda I)(Aw) \ \ (\text{note that } R \text{ is the image of } A - \sigma I : \mathbb{R}^3 \to \mathbb{R}^3)$$

This says that $Av \in R$ also. Hence

$$A : R \to R, \quad \dim R = 2$$

is a linear map and on it we have two eigenvalues $\beta_1,\ \beta_2$ (regardless of what they are). Since we assume $K \not\subset R$, both eigenvalues of A on R must be different from λ, a contradiction. This contradiction implies that $K \subset R$.

Next we claim that $(A - \lambda I)\, R = K$. To see this, note that $(A - \lambda I)\, R$ is **one-dimensional** (it cannot be zero-dimensional since $\dim \ker (A - \lambda I) = 1$) due to $K \subset R$ and the *Rank Theorem* (applied to the map $A - \lambda I : R \to R$). If $(A - \lambda I)\, R \neq K$, there exists some nonzero vector $v \notin K,\ v \in R$, such that $(A - \lambda I)\, R = \{tv : t \in \mathbb{R}\}$. But then we have

$$(A - \lambda I)\, v = tv \quad \text{for some} \quad t \in \mathbb{R}, \quad t \neq 0.$$

This will yield a new eigenvalue $\lambda + t$, impossible. Hence $(A - \lambda I)\, R = K$.

Now let $v_1 \in K$ be an **eigenvector** of A. By above there exists some nonzero vector $v_2 \in R,\ v_2 \notin K$, with (note that $(A - \lambda I)\, R = K$; see also Remark 0.88 below)

$$(A - \lambda I)\, v_2 = v_1.$$

Since $v_2 \in R$, there exists some nonzero vector $v_3 \in \mathbb{R}^3$ such that

$$(A - \lambda I)\, v_3 = v_2 \quad (\text{note that now } (A - \lambda I)^2\, v_3 = v_1).$$

We then claim that $v_1,\ v_2,\ v_3$ are linearly independent. If not, then (we already know that $v_1,\ v_2$ are independent)

$$v_3 = \alpha v_1 + \beta v_2 \quad \text{for some } \alpha,\ \beta.$$

Applying $A - \lambda I$ onto it to get

$$v_2 = (A - \lambda I)\, v_3 = (A - \lambda I)\, (\alpha v_1 + \beta v_2) = \beta\, (A - \lambda I)\, v_2 = \beta v_1,$$

a contradiction. Therefore we have (66) and the proof is done. □

Remark 0.87 *We have the picture for the above proof:*

$$\mathbb{R}^3 \overset{A-\lambda I}{\to} R\ (K \subset R) \overset{A-\lambda I}{\to} K \overset{A-\lambda I}{\to} 0. \tag{86}$$

Remark 0.88 *(**Important**) We claim that it is impossible to have $v_2 \in \mathbb{R}^3,\ v_2 \notin R$, such that*

$$(A - \lambda I)\, v_2 = v_1, \quad \textit{where} \quad v_1 \in K, \quad v_1 \neq 0.$$

To see this, assume possible (note that $v_2 \neq 0$). Then for any $v \in \mathbb{R}^3$ *there exists some vector $\sigma \in R$ (note that* $\dim R = 2$*) such that*

$$v = v_2 + \sigma, \quad v_2 \notin R, \quad \sigma \in R.$$

This implies

$$(A - \lambda I)\, v = (A - \lambda I)\, (v_2 + \sigma) \in K$$

due to $(A - \lambda I)\, v_2 = v_1 \in K$ and the identity $(A - \lambda I)\, R = K$. The above implies $R = K$ (we know $\dim R = 2,\ \dim K = 1$*), a contradiction. In view of this, if we solve the equation*

$$(A - \lambda I)\, v_2 = v_1, \quad \textit{where} \quad v_1 \in K, \quad v_1 \neq 0.$$

then automatically we have $v_2 \in R$. Then one can go directly to find $v_3 \in \mathbb{R}^3$ such that

$$(A - \lambda I)\, v_3 = v_2 \in R.$$

Remark 0.89 *(**Summary.**) In conclusion, we need to solve* v_1 *(eigenvector of* λ, $v_1 \in K$*),* v_2 *(**generalized eigenvector** of* λ *corresponding to* v_1, $v_2 \in R$, $v_2 \notin K$*),* v_3 *(**generalized eigenvector** of* λ *corresponding to* v_2, $v_3 \notin R$*) satisfying the system:*

$$Av_1 = \lambda v_1, \qquad Av_2 = v_1 + \lambda v_2, \qquad Av_3 = v_2 + \lambda v_3. \tag{87}$$

In the third equation of (78), we need to use the fact

$$\begin{cases} K = \ker(A - \lambda I) \subset R = \operatorname{Im}(A - \lambda I), \quad K \text{ is a line and } R \text{ is a plane} \\ (A - \lambda I) R = K \\ v_1 \in K, \qquad v_2 \notin K, \ v_2 \in R, \qquad v_3 \in \mathbb{R}^3. \end{cases} \tag{88}$$

Since $v_2 \in R$*, the equation* $Av_3 = v_2 + \lambda v_3$ *must have a solution for* v_3*.*

Remark 0.90 *In summary, we have the following: Assume* A *is a* 3×3 *real matrix with* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$*, then if* $\ker(A - \lambda I)$ *has three independent eigenvectors (this can happen only when* $A = \lambda I$*), then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

and if $\ker(A - \lambda I)$ *has two independent eigenvectors, then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

and if $\ker(A - \lambda I)$ *has only one independent eigenvector, then*

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Let $P = (v_1, v_2, v_3)$. In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x} = P\mathbf{y}$) becomes the following "**semi-decoupled system**":

$$\frac{d\mathbf{y}}{dt}(t) = \left(P^{-1}AP\right)\mathbf{y} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\mathbf{y}, \quad \mathbf{y} = (y_1, y_2, y_3) \tag{89}$$

and so

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda y_1 + y_2 \\ \dfrac{dy_2}{dt} = \lambda y_2 + y_3 \\ \dfrac{dy_3}{dt} = \lambda y_3. \end{cases}$$

We get (solve $y_3(t)$ first, and then $y_2(t)$, and then $y_1(t)$)

$$y_1(t) = \left(c_1 + c_2 t + \frac{c_3}{2}t^2\right)e^{\lambda t}, \quad y_2(t) = (c_2 + c_3 t)\, e^{\lambda t}, \quad y_3(t) = c_3 e^{\lambda t}$$

and, similar to (83), the general solution to the ODE is given by

$$\mathbf{x}(t) = (v_1, v_2, v_3)\,\mathbf{y}(t) = (v_1, v_2, v_3) \begin{pmatrix} \left(c_1 + c_2 t + \frac{c_3}{2}t^2\right) e^{\lambda t} \\ (c_2 + c_3 t)\, e^{\lambda t} \\ c_3 e^{\lambda t} \end{pmatrix}$$

$$= \left(c_1 + c_2 t + \frac{c_3}{2}t^2\right) e^{\lambda t} v_1 + (c_2 + c_3 t)\, e^{\lambda t} v_2 + c_3 e^{\lambda t} v_3. \tag{90}$$

Note that in the above only v_1 is eigenvector.

Example 0.91 *Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solution:

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix}$$
$$= (1-\lambda)^2 (4-\lambda) + 4 + 3 + 3(1-\lambda) + 2(1-\lambda) - 2(4-\lambda)$$
$$= \left(\lambda^2 - 2\lambda + 1\right)(4-\lambda) + 4 - 3\lambda$$
$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3.$$

Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = 2$. To find the eigenvector for $\lambda = 2$, we solve

$$\begin{cases} x + y + z = 2x \\ 2x + y - z = 2y \\ -3x + 2y + 4z = 2z \end{cases}$$

and we obtain $x = 0,\ y + z = 0$. Thus we can find **only one** independent eigenvector $v_1 = (0, 1, -1)$. The image of the matrix

$$A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

is the plane R given by $x - y - z = 0$ (or the plane spanned by the two vectors $(-1, 2, -3)$, $(1, -1, 2)$).

Then we solve $Av_2 = v_1 + 2v_2$ to get

$$\begin{cases} x + y + z = 2x \\ 2x + y - z = 1 + 2y \\ -3x + 2y + 4z = -1 + 2z \end{cases}$$

and obtain $x = 1,\ y + z = 1$. We can pick $v_2 = (1, 1, 0)$ (by Remark 0.88 we must have $v_2 \in R,\ v_2 \notin K$, or one can check that v_2 lies in the plane $x - y - z = 0$, or $v_2 = 2(-1, 2, -3) + 3(1, -1, 2)$). Finally, we solve $Av_3 = v_2 + 2v_3$ to get

$$\begin{cases} x + y + z = 1 + 2x \\ 2x + y - z = 1 + 2y \\ -3x + 2y + 4z = 2z \end{cases}$$

and obtain $x = 2,\ y + z = 3$. We can pick $v_3 = (2, 3, 0)$. Hence the general solution is given by

$$\mathbf{x}(t) = \left(c_1 + c_2 t + \frac{c_3}{2}t^2\right) e^{2t} v_1 + (c_2 + c_3 t)\, e^{2t} v_2 + c_3 e^{2t} v_3 = \ \ldots\ldots$$

□

0.5.4 $\lambda_1 = \lambda,\ \lambda_2 = \alpha + i\beta,\ \lambda_3 = \alpha - i\beta.$

Assume we have three eigenvalues $\lambda \in \mathbb{R}$ and $\alpha + i\beta,\ \alpha - i\beta,\ \alpha,\ \beta \in \mathbb{R},\ \beta \neq 0$. There exists a basis $\{v, v_1, v_2\}$ satisfying (see (45)) (now the eigenvector of $\alpha + i\beta$ is $v_2 + iv_1$)

$$Av = \lambda v, \quad Av_1 = \alpha v_1 + \beta v_2, \quad Av_2 = -\beta v_1 + \alpha v_2,$$

and so

$$AP = P\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{pmatrix}, \quad P = (v, v_1, v_2).$$

In this case the ODE for $\mathbf{y}(t)$ (we let $\mathbf{x} = P\mathbf{y}$) becomes the following "**semi-decoupled system**":

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda y_1 \\ \dfrac{dy_2}{dt} = \alpha y_2 - \beta y_3 \\ \dfrac{dy_3}{dt} = \beta y_2 + \alpha y_3 \end{cases} \tag{91}$$

and its general solution is given by

$$\begin{cases} y_1(t) = c_1 e^{\lambda t} \\ y_2(t) = e^{\alpha t}(c_2 \cos\beta t - c_3 \sin\beta t) \\ y_3(t) = e^{\alpha t}(c_2 \sin\beta t + c_3 \cos\beta t). \end{cases} \tag{92}$$

Hence the general solution $\mathbf{x}(t)$ to the ODE $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\begin{aligned} \mathbf{x}(t) = P\mathbf{y}(t) &= (v, v_1, v_2)\begin{pmatrix} c_1 e^{\lambda t} \\ e^{\alpha t}(c_2 \cos\beta t - c_3 \sin\beta t) \\ e^{\alpha t}(c_2 \sin\beta t + c_3 \cos\beta t) \end{pmatrix} \\ &= c_1 e^{\lambda t} v + c_2 e^{\alpha t}\left[(\cos\beta t)\, v_1 + (\sin\beta t)\, v_2\right] + c_3 e^{\alpha t}\left[-(\sin\beta t)\, v_1 + (\cos\beta t)\, v_2\right]. \end{aligned} \tag{93}$$

Example 0.92 *Find the general solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

<u>**Solution:**</u>

The characteristic polynomial of the matrix is of the coefficient matrix are

$$\begin{vmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} = -(\lambda + 2)\left(\lambda^2 + 2\lambda + 3\right).$$

Hence the eigenvalues are $\lambda_1 = -2,\ \lambda_2 = -1 + \sqrt{2}i,\ \lambda_3 = -1 - \sqrt{2}i$. To find the eigenvector for $\lambda = -2$, we solve

$$\begin{cases} -3x + 2z = -2x \\ x - y = -2y \\ -2x - y = -2z \end{cases}$$

and we obtain $x = 2z,\ y = -2z$. Thus $v = (2, -2, 1)$. To find the eigenvector for $\lambda = -1 + \sqrt{2}i$, we solve

$$\begin{cases} -3x + 2z = \left(-1 + \sqrt{2}i\right) x \\ x - y = \left(-1 + \sqrt{2}i\right) y \\ -2x - y = \left(-1 + \sqrt{2}i\right) z \end{cases}$$

and get complex eigenvector

$$u = \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 + \sqrt{2}i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}.$$

So we get $v_1 = \left(\sqrt{2}, 0, \sqrt{2}\right)$ and $v_2 = (0, 1, -1)$. The general solution is given by

$$\mathbf{x}(t) = c_1 e^{-2t} v + c_2 e^{-t} \left[\left(\cos\sqrt{2}t\right) v_1 + \left(\sin\sqrt{2}t\right) v_2\right] + c_3 e^{-t} \left[-\left(\sin\sqrt{2}t\right) v_1 + \left(\cos\sqrt{2}t\right) v_2\right].$$

□