Shreve 4.5 Black-Scholes-Merton Equation

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- Derive the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as a geometric Brownian motion.

Portfolio Value

Portfolio $X(t)$

Consider a portfolio at time t valued at X(t), which invests in:

- Money market:

  Pay a constant rate of interest r

- Stock:

  Stock price modeled by the geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

The investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$, $t \geq 0$.

$$\begin{aligned}
dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\
&= \Delta(t)(\alpha S(t)dt + \sigma S(t)dW(t)) \\
&\quad + r(X(t) - \Delta(t)S(t))dt \\
&= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt \\
&\quad + \Delta(t)\sigma S(t)dW(t)
\end{aligned}$$

The 3 terms appearing in $dX(t)$ can be understood as follows:

1. an average underlying rate of return r on the portfolio.
2. a risk premium $\alpha - r$ for investing in the stock.
3. a volatility term proportional to the size of the stock investment.

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

Consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value $e^{-rt}X(t)$.

The discounted stock price $e^{-rt}S(t)$

According to the Itô-Doeblin formula with $f(t,x) = e^{-rt}x$:

$$\begin{aligned} de^{-rt}S(t) &= df(t, S(t)) \\ &= f_t(t, S(t))dt + f_x(t, S(t))dS(t) \\ &\quad + \frac{1}{2} f_{xx}(t, S(t))dS(t)^2 \\ &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= (\alpha - r)e^{-rt}dt + \sigma e^{-rt}S(t)dW(t) \end{aligned}$$

The discounted portfolio value $e^{-rt}X(t)$

$$\begin{aligned}
de^{-rt}X(t) &= df(t, X(t)) \\
&= f_t(t, X(t))dt + f_x(t, X(t))dX(t) \\
&\quad + \frac{1}{2} f_{xx}(t, X(t))dX(t)^2 \\
&= \ - re^{-rt}X(t)dt + e^{-rt}dX(t) \\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \\
&= \Delta(t)d(e^{-rt}S(t))
\end{aligned}$$

The change in the discounted portfolio value is solely due to change in the discounted stock price.

Option Value

- Consider a European call option that pays $(S(T)-K)^+$ at time T.
- we let $c(t,x)$ denote the value of the call at time t if the stock price at that time is $S(t)=x$

The differential of $c(t, S(t))$

$$
\begin{aligned}
& dc(t, S(t)) \\
&= c_t(t, S(t))dt + c_x(t, S(t))dS(t) \\
&\quad + \frac{1}{2}c_{xx}(t, S(t))dS(t)^2 \\
&= c_t(t, S(t))dt + c_x(t, S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) \\
&\quad + \frac{1}{2}c_{xx}(t, S(t))\sigma^2 S^2(t)dt \\
&= \left[c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}c_{xx}(t, S(t))\sigma^2 S^2(t)\right] dt \\
&\quad + \sigma S(t)c_x(t, S(t))dW(t)
\end{aligned}
$$

The differential of discounted option price $c(t, S(t))$

$$
\begin{aligned}
& d(e^{-rt}c(t,S(t))) \\
&= df(t, c(t,S(t))) \\
&= f_t(t,c(t,S(t)))dt + f_x(t,c(t,S(t)))dc(t,S(t)) \\
&\quad + \frac{1}{2} f_{xx}(t,c(t,S(t)))dc(t,S(t))^2 \\
&= -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)) \\
&= e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\
&\quad + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)\Big]dt + e^{-rt}\sigma S(t)c_x(t,S(t))dW(t)
\end{aligned}
$$

Equating the Evolutions

- A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account.
- The portfolio value $X(t), \quad \forall t \in [0, T]$ agrees with $c(t, S(t))$ if and only if $e^{-rt}X(t) = e^{-rt}c(t, S(t)), \quad \forall t \in [0, T]$.

- To ensure this equality we should make sure that

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t))), \quad \forall t \in [0, T)$$

and $X(0) = c(0, S(0))$.

- Integration of above function from 0 to t

$$e^{-rt}X(t) - X(0) = e^{-rt}c(t, S(t)) - c(0, S(0))$$

$$\Rightarrow e^{-rt}X(t) = e^{-rt}c(t, S(t)), \quad \forall t \in [0, T)$$

Comparing $de^{-rt}X(t)$ and $de^{-rt}c(t, S(t))$

$$\begin{aligned}
&\Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \\
&= e^{-rt}\Big[-rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) \\
&\quad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))\Big] dt + e^{-rt}\sigma S(t)c_x(t, S(t))dW(t)
\end{aligned}$$

We first equate the $dW(t)$ terms, which gives :

$$\Delta(t) = c_x(t, S(t)), \quad \forall t \in [0, T)$$

This is called the delta-hedging rule.

Comparing $de^{-rt}X(t)$ and $de^{-rt}c(t,S(t))$

$$\Delta(t)(\alpha-r)e^{-rt}S(t)dt+\Delta(t)\sigma e^{-rt}S(t)dW(t)$$
$$=e^{-rt}\Big[-rc(t,S(t))+c_t(t,S(t))+\alpha S(t)c_x(t,S(t))$$
$$+\frac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\Big]dt+e^{-rt}\sigma S(t)c_x(t,S(t))dW(t)$$

Next equate the dt terms

$$\Delta(t)(\alpha-r)e^{-rt}S(t)=e^{-rt}\Big[-rc(t,S(t))+c_t(t,S(t))$$
$$+\alpha S(t)c_x(t,S(t))+\tfrac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\Big]$$

$$\Delta(t)(\alpha - r)S(t) = \Big[- rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))\Big]$$

where $\Delta(t)\alpha S(t) = \alpha S(t)c_x(t, S(t))$

$$\Rightarrow rc(t,S(t)) = c_t(t, S(t)) + rS(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))$$

- In conclusion, we should seek a continuous function $c(t, x)$ that is a solution to the Black-Scholes-Merton partial differential equation.

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$$

$$\forall t \in [0, T), \quad x \geq 0$$

- and that satisfies the terminal condition

$$c(T,x) = (x-K)^+$$

- If an investor starts with initial capital $X(0)$ and uses the hedge $\Delta(t) = c_x(t, S(t))$
- then $d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t)))$ will hold.
- We see that $X(t) = c(t, S(t))$ for all $t \in [0, T)$
- Taking the limit as $t \to T$ and using the fact that both $X(t)$ and $c(t, S(t))$ are continuous, we conclude that
  $X(T) = c(T, S(T)) = (S(T) - K)^+$
- This means that the short position has been successfully hedged.

Solution of B-S-M Equation

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$$
$$\forall t \in [0,T), \quad x \geq 0$$

- The Black-Scholes-Merton equation doesn't involve probability, it is a partial differential equation.

Backward parabolic

Black-Scholes-Merton PDE is a PDE of the type called Backward parabolic

$$Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_x + Eu_t + F = 0$$

and this function needs :

1. $B^2 - AC = 0$
2. $\partial\Omega \times (0, T)$ boundary condition
3. $\Omega \times \{T\}$ terminal condition

For such an equation, in addition to the terminal condition $c(T,x) = (S(T) - K)^+$, one needs boundary conditions at $x = 0$ and $x = \infty$ in order to determine the solution.

1. Substituting $x = 0$ into Black-Scholes-Merton PDE, which then becomes

$$c_t(t, 0) = rc(t, 0)$$

and the solution is

$$c(t, 0) = e^{rt} c(0, 0)$$

Substituting $t = T$ into this equation and using the fact that $c(T, 0) = (0 - K)^+ = 0$, we see that $c(0, 0) = 0$ and hence

$$c(t, 0) = 0, \quad \forall t \in [0, T]$$

this is the boundary condition at x=0.

2. As $x \to \infty$,the function $c(t,x)$ grows without bound. One way to specify a boundary condition at $x = \infty$ for the European call is

$$\lim_{x\to\infty}\left[c(t,x) - (x - e^{-r(T-t)}K)\right] = 0, \quad \forall t \in [0,T]$$

For large x, this call is deep in the money and very likely to end in the money.

Black-Scholes-Merton function

The solution to the Black-Scholes-Merton equation with terminal condition and boundary conditions is

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

$$\forall t \in [0,T), \quad x > 0$$

where

$$d_\pm(\tau,x) = \frac{ln(\frac{x}{K}) + (r \pm \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}dz$$

but above function does not define $c(t,x)$ when $t=T$,nor dose it define $c(t,x)$ when $x=0$.

$$\lim_{t\to T} c(t,x) = (x-K)^+$$

$$\lim_{x\to 0} c(t,x) = 0$$

Black-Scholes-Merton function

We shall sometimes use the notation

$$BSM(\tau,x;K,r,\sigma) = xN(d_+(\tau,x)) - Ke^{-r(T-t)}N(d_-(\tau,x))$$

The Greeks

- The derivatives of the function $c(t, x)$ with respect to various variables are called the Greeks.

Delta	Δ	$\partial c / \partial x$
Gamma	γ	$\partial^2 c / \partial x^2$
Theta	θ	$\partial c / \partial t$
Vega	ν	$\partial c / \partial \sigma$
Rho	ρ	$\partial c / \partial r$

- Delta is always positive

$$\Delta = c_x(t,x) = N(d_+(T-t,x))$$

- Gamma is always positive

$$\gamma = c_{xx}(t,x) = \frac{N'(d_+(T-t,x))}{\sigma x\sqrt{T-t}}$$

- Theta is always negative

$$\begin{aligned}\theta = c_t(t,x) = & -rKe^{-r(T-t)}N(d_-(T-t,x)) \\ & -\frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))\end{aligned}$$

Because delta and gamma are positive, for fixed t , the function $c(t, x)$ is increasing and convex in the variable x.

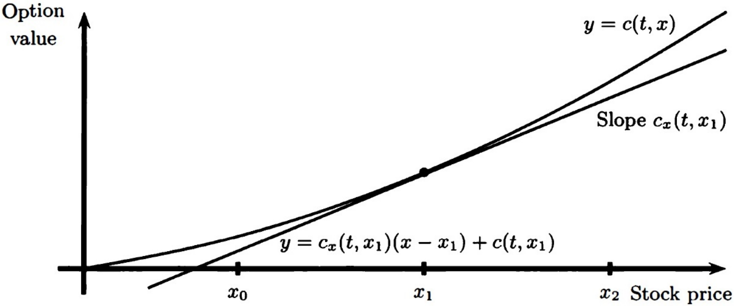

Fig. 4.5.1. Delta-neutral position.

- If at time t the stock price is x, then the short option hedge of calls for holding $c_x(t,x)$ shares of stock.
- The hedging portfolio value is

$$c = xN[d_+] - Ke^{-r(T-t)}N[d_-]$$

- The amount invested in the money market is

$$c(t,x) - xc_x(t,x) = -Ke^{-r(T-t)}N[d_-] < 0$$

Consider a portfolio when stock price is x_1 and we wish to take a long position in the option and hedge it.

1. Long the option for $c(t, x_1)$
2. Short $c_x(t, x_1)$ shares of stock
3. Invest in money market account M

$$M = x_1 c_x(t, x_1) - c(t, x_1)$$

The initial portfolio value

$$c(t, x_1) - x_1 c_x(t, x_1) + M = 0$$

- If the stock price were to instantaneously fall to x_0 and we do not change our portfolio.
- Total portfolio value would be

$$
\begin{aligned}
&c(t,x_0) - x_0 c_x(t,x_1) + M \\
&\quad = c(t,x_0) - c_x(t,x_1)(x_0 - x_1) - c(t,x_1)
\end{aligned}
$$

- This is the difference at x_0 between the curve $y = c(t,x)$ and the straight line $y = c_x(t,x_1)(x - x_1) + c(t,x_1)$

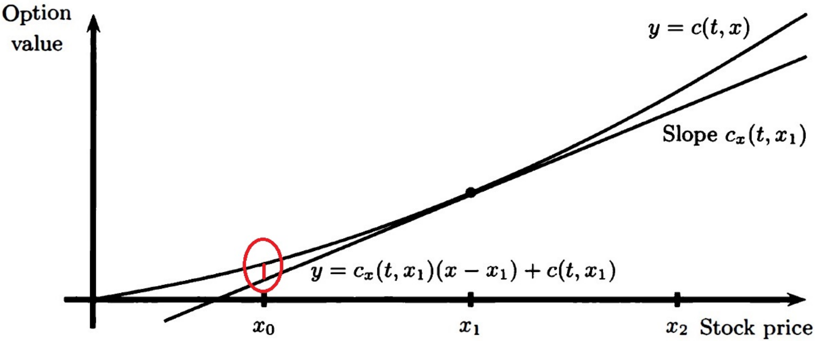

Fig. 4.5.1. Delta-neutral position.

- The portfolio we have set up is said to be delta-neutral and long gamma

Long Gamma

The portfolio is long gamma because it benefits from the convexity of $c(t, x)$

1. If the stock price instantaneously falls or rises
2. The value of the portfolio increases
3. A long gamma portfolio is profitable in times of high stock volatility

- The portfolio described above may at first appear to offer an arbitrage opportunity.
  Is above argument correct ?
- When we let time move forward, the curve $c(t, x)$ is shifts downward because theta θ is negative.
- To keep the portfolio delta-neutral, we have to continuously rebalance our portfolio.

- Actually, assets are not really geometric Brownian motions with constant volatility.
- The derivative of the option price with respect to the volatility σ is called Vega.
- Vega is positive, as volatility increases, option prices in the Black-Scholes-Merton model would increase.

Put-Call Parity

- A forward contract with delivery price K obligates its holder to buy one share of the stock at the expiration time T in exchange for payment K
- At expiration, the value of the forward contract is $S(T) - K$
- The value of the forward contract :

$$
\begin{aligned}
f(t, S(t)) &= E^{Q}\left[e^{-r(T-t)}(S(T) - K) | F_t\right] \\
&= S(t) - e^{-r(T-t)}K, \quad \forall t \in [0, T]
\end{aligned}
$$

- The agent sells this forward contract at $t = 0$ for $f(t, S(0)) = S(0) - e^{-rT}K$
- He can set up a static hedge, in order to protect himself.

Static Hedge

A hedge that does not trade except at the initial time

- He should purchase one share of stock by initial capital from the sale of the forward contract $S(0) - e^{-rT}K$ and the money $e^{-rT}K$ borrow from money market account.

Forward price of $S(t)$

- The forward price of a stock at time t is defined to be the value of K that cause the forward contract at time t to have value zero
- The forward price at time t :

$$For(t) = e^{r(T-t)} S(t)$$

- The forward price at time t is the price one can lock in at time t for the purchase of one share of stock at time T, paying the price at time T

- Consider a situation at $t = 0$, one can lock in a price $For(0) = e^{rT}S(0)$ for buying a stock at time T
- The value of the forward contract would be rewritten as

$$f(t, S(t)) = S(t) - e^{rT}S(0)$$

We observe that for any number x, the equation

$$x - K = (x - K)^+ - (K - x)^+$$

The payoff of European Options at $t = T$

1. European call $c(T, S(T))$: $(S(T) - K)^+$
2. European put $p(T, S(T))$: $(K - S(T))^+$
3. Forward contract $f(T, S(T))$: $S(T) - K$

- The equation $x - K = (x - K)^+ - (K - x)^+$ implies

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T))$$

- The payoff of the forward contract agrees with the payoff of a portfolio that is long a call and short a put.
- These values must agree at all previous times

$$f(t, x) = c(t, x) - p(t, x), \quad \forall x \geq 0, 0 \leq t \leq T$$

Put-Call Parity

The relationship of equation

$$f(t,x) = c(t,x) - p(t,x)$$

is called Put-Call Parity.
The equation could also be denoted as

$$S - e^{-rT}K = C - P$$

Black-Scholes-Merton put option formula

Finally, we can use Put-Call Parity and Black-Scholes-Merton call option formula to obtain the put option formula.

$$\begin{aligned} p(t,x) =& xN(d_+(T-t,x)) - x \\ & - Ke^{-r(T-t)}N(d_-(T-t,x)) + e^{-r(T-t)}K \\ =& Ke^{-r(T-t)}N(-d_-(T-t,x)) \\ & - xN(-d_+(T-t,x)) \end{aligned}$$

Thank you !