3.7 Reflection Principle

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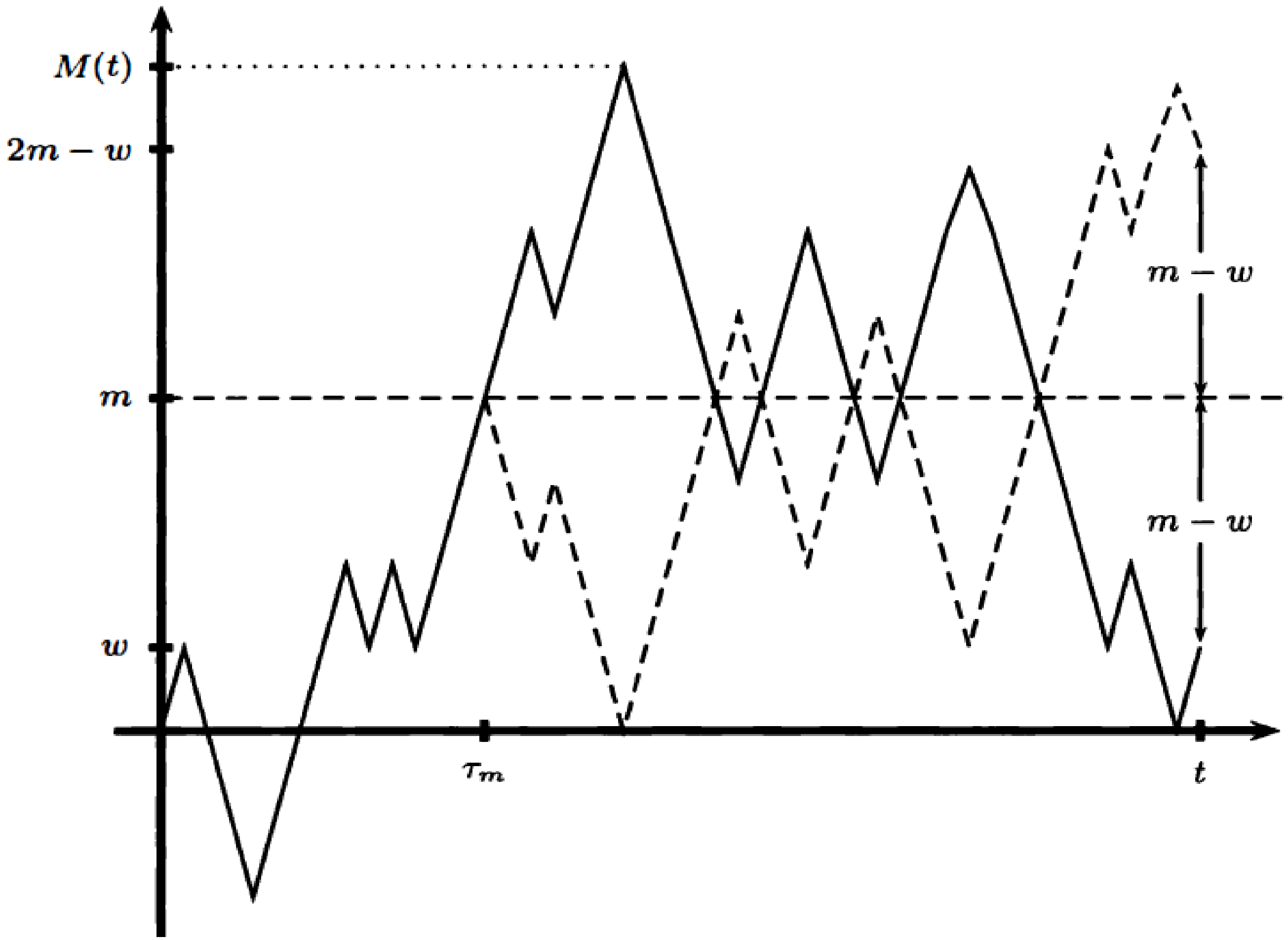
3.7.1 Reflection Equality



$$P\{\tau_m \leq t, W(t) \leq \omega\} = P\{W(t) \geq 2m - \omega\}, \qquad \omega \leq m\,, m > 0$$

3.7.2 First Passage Time Distribution

- Thm 3.7.1
- $\forall m \neq 0$, the random variable τ_m has cumulative distribution function

$$P\{\tau_m \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, \qquad t \geq 0$$

and density

$$f_{\tau_m}(t) = \frac{d}{dt} P\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, \qquad t \geq 0$$

Thm 3.7.1 Proof (1)

- Let $m > 0$. We substitute $\omega = m$ into the reflection formula

$$P\{\tau_m \leq t, W(t) \leq \omega\} = P\{W(t) \geq 2m - \omega\} \xrightarrow{\omega = m}$$

$$P\{\tau_m \leq t, W(t) \leq m\} = P\{W(t) \geq m\}$$

- if $W(t) \geq m$, then we are guaranteed that $\tau_m \leq t$

  so $P\{\tau_m \leq t, W(t) \geq m\} = P\{W(t) \geq m\}$

- $P\{\tau_m \leq t\} = P\{\tau_m \leq t, W(t) \leq m\} + P\{\tau_m \leq t, W(t) \geq m\}$

$$= 2P\{W(t) \geq m\} = \frac{2}{\sqrt{2\pi t}} \int_m^{\infty} e^{-\frac{x^2}{2t}} dx$$

Thm 3.7.1 Proof (2)

- Make the change of variable

$$\frac{2}{\sqrt{2\pi t}}\int_{m}^{\infty} e^{-\frac{x^2}{2t}}dx \xrightarrow{y=\frac{x}{\sqrt{t}}} \frac{2}{\sqrt{2\pi}}\int_{\frac{m}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}dy$$

- If m is negative, then $\tau_m$and $\tau_{|m|}$have the same distribution

$$\frac{2}{\sqrt{2\pi}}\int_{\frac{m}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}dy$$

differentiate to obtain

$$f_{\tau_m}(t) = \frac{d}{dt}P\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}}e^{-\frac{m^2}{2t}}, \qquad t \geq 0$$

Remark 3.7.2 Another Representation of Laplace Transform of First Passage Time

$$E[e^{-\alpha\tau_m}] = \int_0^\infty e^{-\alpha t} f_{\tau_m}(t)dt = \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\alpha t - \frac{m^2}{2t}} dt$$

$$= e^{-|m|\sqrt{2\alpha}}, \qquad for\ all\ \alpha > 0$$

3.7.3 Distribution of Brownian motion and its Maximum

- Define the maximum to date for Brownain motion to be

$$M(t) = \max_{0 \le s \le t} W(s)$$

- For positive m, we have $M(t) \geq m, \quad iff\ \tau_m \leq t$

$$P\{M(t) \geq m, W(t) \leq \omega\} = P\{W(t) \geq 2m - \omega\}, \qquad \omega \leq m, m > 0$$

The Joint Distribution of $\boldsymbol{W(t)}$and $\boldsymbol{M(t)}$

- Thm 3.7.3
- For $t > 0$, the joint density of $(M(t),\ W(t))$ is

$$f_{M(t),W(t)}(m,\omega) = \frac{2(2m-\omega)}{t\sqrt{2\pi t}} e^{-\frac{(2m-\omega)^2}{2t}}, \qquad \omega \le m, m > 0$$

Proof:

$$\because P\{M(t) \ge m, W(t) \le \omega\} = \int_m^\infty \int_{-\infty}^{\omega} f_{M(t),W(t)}(x,y)dy\,dx$$

$$P\{W(t) \ge 2m-\omega\} = \frac{1}{\sqrt{2\pi t}} \int_{2m-\omega}^{\infty} e^{-\frac{z^2}{2t}} dz$$

Thm 3.7.3 Proof (continue)

$$\int_m^{\infty}\int_{-\infty}^{\omega} f_{M(t),W(t)}(x,y)dy\,dx = \frac{1}{\sqrt{2\pi t}}\int_{2m-\omega}^{\infty} e^{-\frac{z^2}{2t}}dz$$

Differentiate with respect to m to obtain

$$-\int_{-\infty}^{\omega} f_{M(t),W(t)}(m,y)dy = -\frac{2}{\sqrt{2\pi t}}e^{-\frac{(2m-\omega)^2}{2t}}$$

Next differentiate with respect to ω to see that

$$f_{M(t),W(t)}(m,\omega) = \frac{2(2m-\omega)}{t\sqrt{2\pi t}}e^{-\frac{(2m-\omega)^2}{2t}}$$

Corollary 3.7.4

- The conditional distribution of $M(t)$ given is $W(t) = \omega$

$$f_{M(t)|W(t)}(m|\omega) = \frac{2(2m-\omega)}{t} e^{-\frac{2m(m-\omega)}{t}}, \qquad \omega \le m, m > 0$$

- Proof:

$$f_{M(t)|W(t)}(m|\omega) = \frac{f_{M(t),W(t)}(m,\omega)}{f_{W(t)}(\omega)}$$

$$= \frac{2(2m-\omega)}{t\sqrt{2\pi t}} \cdot \sqrt{2\pi t} \cdot e^{-\frac{(2m-\omega)^2}{2t}+\frac{\omega^2}{2t}}$$

$$= \frac{2(2m-\omega)}{t} \cdot e^{-\frac{2m(m-\omega)}{t}}$$

THE END