3.6 First Passage Time Distribution

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Theorem 3.6.1

- Exponential Martingale
- Let $W(t), t \geq 0$, be a Brownain motion with a filtration $F(t), t \geq 0$, and let σ be a constant. The process $Z(t), t \geq 0$, is a martingale.

$$Z(t) = e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$$

Theorem 3.6.1 Proof (1)

- For $0 \le s \le t$, we have

$$
\begin{aligned}
E[Z(t)|F(s)] &= E\left[e^{\sigma W(t)-\frac{1}{2}\sigma^2 t}\middle|F(s)\right] \\
&= E\left[e^{\sigma W(t)-\sigma W(s)+\sigma W(s)-\frac{1}{2}\sigma^2 t}\middle|F(s)\right] \\
&= E\left[e^{\sigma(W(t)-W(s))}\cdot e^{\sigma W(s)-\frac{1}{2}\sigma^2 t}\middle|F(s)\right] \\
&= e^{\sigma W(s)-\frac{1}{2}\sigma^2 t}\cdot E\left[e^{\sigma(W(t)-W(s))}\middle|F(s)\right] \\
&= e^{\sigma W(s)-\frac{1}{2}\sigma^2 t}\cdot E\left[e^{\sigma(W(t)-W(s))}\right]
\end{aligned}
$$

Theorem 3.6.1 Proof (2)

- Next we know $W(t) - W(s)$ is normally destibution with mean $\mathrm{E}[W(t) - W(s)] = 0$ and variance $\sigma^2 = t - s$
- By 3.2.13

The moment generating function of a normal distribution $N(0, t)$

$$\phi(u) = E[e^{uX}] = \int_{-\infty}^{\infty} e^{ux} f(x) dx = e^{\frac{1}{2}u^2 t}$$

$$E\left[e^{\sigma\left(W(t)-W(s)\right)}\right] = e^{\frac{1}{2}\sigma^2(t-s)}$$

- Such that $E[Z(t)|F(s)] = e^{\sigma W(s)-\frac{1}{2}\sigma^2 t} \cdot E\left[e^{\sigma(W(t)-W(s))}\right]$

$$= e^{\sigma W(s)-\frac{1}{2}\sigma^2 t} \cdot e^{\frac{1}{2}\sigma^2(t-s)}$$

$$= e^{\sigma W(s)-\frac{1}{2}\sigma^2 s} = Z(s)$$

Definition of First Passage Time

- Let $m \in R$, and define the first passage time to level m

$$\tau_m = \min\{t \geq 0 | W(t) = m\}$$

This is the first time the Brownian motion W reaches the level m.

- If the Brownian motion never reaches the level m, we set $\tau_m \to \infty$
- By Theorem 4.3.2 of volume I (Optional Sampling Thm)
  - A martingale that is stopped at a stopping time is still a martingale and thus must have constant expectation
- $1 = Z(0) = E[Z(t \wedge \tau_m)] = E[e^{\sigma W(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)}]$

where the notation $t \wedge \tau_m$ denotes the minimum of t and τ_m

Properties of First Passage Time

- Let $\sigma > 0$ and $m > 0$. The Brownian motion is always at or below level m for $t < \tau_m$ and so

$$0 \leq e^{\sigma w(t \wedge \tau_m)} \leq e^{\sigma m}$$

- The goal is to show that probability to hit = 1

$$P(\tau_m < \infty) = 1$$

And

$$E\left[e^{-\frac{1}{2}\sigma^2 \tau_m}\right] = e^{-\sigma m}$$

The Property of $e^{\sigma W(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)}$

	$\tau_m < \infty$	$\tau_m = \infty$
$e^{-\frac{1}{2}\sigma^2 (t \wedge \tau_m)}$	$e^{-\frac{1}{2}\sigma^2 \tau_m}$	0
$e^{\sigma W(t \wedge \tau_m)}$	$e^{\sigma m}$	$0 \leq e^{\sigma w(t \wedge \tau_m)} \leq e^{\sigma m}$
$Z(t \wedge \tau_m)$	$e^{\sigma m - \frac{1}{2}\sigma^2 \tau_m}$	0

The Property of $e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}$

- As t become large enough, if $\tau_m < \infty$, then the term
$$e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} = e^{-\frac{1}{2}\sigma^2\tau_m}$$
- If $\tau_m = \infty$, the term
$$e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} = e^{-\frac{1}{2}\sigma^2 t}$$

  and as $t \to \infty$, this converge to zero
- We capture these two cases by writing
$$\lim_{t\to\infty} e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} = I_{(\tau_m<\infty)} e^{-\frac{1}{2}\sigma^2\tau_m}$$

The Property of $e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}$

- As t become large enough, if $\tau_m < \infty$, then the term

$$e^{\sigma W(t\wedge\tau_m)} = e^{\sigma W(\tau_m)} = e^{\sigma m}$$

- If $\tau_m = \infty$, $e^{\sigma W(t\wedge\tau_m)}$ is bounded because of

$$0 \le e^{\sigma w(t\wedge\tau_m)} \le e^{\sigma m}$$

that is enough to ensure that

$$\begin{aligned} 0 &\le \lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)} \\ &\le e^{\sigma m} \lim_{t\to\infty} e^{-\frac{1}{2}\sigma^2 t} = 0 \end{aligned}$$

Derivation from $\lim_{t\to\infty} E[e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}]$

- In conclusion, we have

$$\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)} = I_{(\tau_m<\infty)}e^{\sigma m-\frac{1}{2}\sigma^2\tau_m}$$

- Recall the martingale property:

$$1 = Z(0) = E[Z(t\wedge\tau_m)] = E[e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}]$$

- Take $t\to\infty$(interchange of limit and expectation, by Dominated Convergence Theorem: Dominated by $e^{\sigma m}$)

$$1 = E\left[\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m)-\frac{1}{2}\sigma^2(t\wedge\tau_m)}\right] = \mathrm{E}\left[I_{(\tau_m<\infty)}e^{\sigma m-\frac{1}{2}\sigma^2\tau_m}\right]$$

or, equivalently,

$$E\left[I_{(\tau_m<\infty)}e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$$

hold when $m > 0$ and $\sigma > 0$.

Obtain the Result : $P\{\tau_m < \infty\} = 1$

- The below holds for every positive σ, take the limit $\sigma \to 0$

  because $W(t)$ is not relative on σ

$$E\left[I_{(\tau_m<\infty)}e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$$

- This yields $E\left[I_{(\tau_m<\infty)}\right] = 1$ (using the Monotone Convergence Theorem), or equivalently,

$$P\{\tau_m < \infty\} = 1$$

- We can think this equation from $E[f(X)e^{tX}] = g(t)$

  Our target is $E[f(X)]$, so we assume $t = 0$

$$E[f(X)] = g(0)$$

Obtain the Result : $E\left[e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$

- Because τ_m is finite with probability 1 almost surely, we may drop the indicator of this event

$$E\left[I_{(\tau_m<\infty)}e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$$

and obtain

$$E\left[e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$$

Thm 3.6.2 Laplace Transform of First Passage Time Distribution

- For $m \in R$, the first passage time of Brownian motion to level m is finite almost surely.
- The Laplace transform of its distribution is given by

$$E[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}, \qquad for\ all\ \alpha > 0$$

Proof of Theorem 3.6.2

- Recall $E\left[e^{-\frac{1}{2}\sigma^2\tau_m}\right] = e^{-\sigma m}$
- When m is positive. Set $\sigma = \sqrt{2\alpha}$, so that $\frac{1}{2}\sigma^2 = \alpha$.
- If m is negative, then because Brownian motion is symmetric, the first passage times τ_m and $\tau_{|m|}$have same distribution.
- Equation

$$E[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}, \qquad for\ all\ \alpha > 0$$

for negative m follows.

Remark: $P(\tau_m < \infty) = 1$, but $E[\tau_m] = \infty$

- Differentiation of

$$E[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}, \qquad for\ all\ \alpha > 0$$

with repect to α results in

$$E[\tau_m e^{-\alpha\tau_m}] = \frac{|m|}{\sqrt{2\alpha}} e^{-|m|\sqrt{2\alpha}}, \qquad for\ all\ \alpha > 0$$

- Letting $\alpha \to 0$, we obtain $E[\tau_m] = \infty$ so long as $m \neq 0$.
  - Using Monotone Convergence Theorem

Laplace Transform

$$\begin{array}{ccc} f(t) & \longrightarrow & g(t) \\ \downarrow & & \uparrow \\ F(s) & \longrightarrow & G(s) \end{array}$$

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t)dt$$

$$L^{-1}(G) = \frac{1}{2\pi i} \lim_{\mathrm{T}\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} G(s)ds$$

- Laplace Transform usually be used to solve some difficult integration problem or differential equation.

Laplace Transform : Example of Application

- Moment generating function

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} f_X(x)dx$$

- Cumulative distribution function

$$P(X \leq x) = \int_{-\infty}^{x} f_X(y)dy = L^{-1}\left(\frac{1}{s}Lf\ (s)\right)$$

Cumulative distribution function

proof:

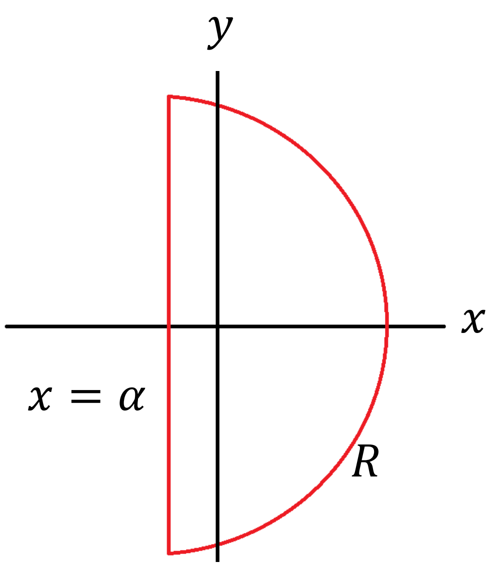
$$L(f) = \int_{-\infty}^{\infty} e^{-st} f(t)dt = \int_{-\infty}^{\infty} e^{-st} dF(t)$$

$$= e^{-st} F(t)\Big|_{t=-\infty}^{t=\infty} + s\int_{-\infty}^{\infty} e^{-st} F(t)dt = s\int_{-\infty}^{\infty} e^{-st} F(t)dt$$

$$s.t.\ \frac{1}{s} Lf(t) = \int_{-\infty}^{\infty} e^{-st} F(t)dt = L(F)$$

$$L^{-1}\left(\frac{1}{s} L(f)\right) = F(t) = \int_{-\infty}^{t} f_X(x)dx$$

Example of Laplace transform



- Let $f(t) = 1$ and its Laplace transform is :

$$L(f) = \int_0^\infty e^{-st} \cdot 1 dt = \frac{1}{s}$$

- Now, we use Laplace inverse transform and the integral path is shown by upon figture.

$$L^{-1}\left(\frac{1}{s}\right) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\alpha-iT}^{\alpha+iT} e^{st} \cdot \frac{1}{s} ds$$

$$= \frac{1}{2\pi i} \oint e^{st} \cdot \frac{1}{s} ds - \int_{C_{R\to\infty}} e^{st} \cdot \frac{1}{s} ds$$

Example of Laplace transform

$$And \int_{C_{R\to\infty}} e^{st}\cdot\frac{1}{s}ds \; would\ approach\ to\ 0\ by\ complex\ integration$$

$$s.t. \qquad L^{-1}\left(\frac{1}{s}\right)=\frac{1}{2\pi i}\oint e^{st}\cdot\frac{1}{s}ds$$

$$Next\ we\ use\ Cauchy's\ integral\ formula$$

$$L^{-1}\left(\frac{1}{s}\right)=\frac{1}{2\pi i}\oint e^{st}\cdot\frac{1}{s}ds=\frac{1}{2\pi i}\cdot 2\pi i\cdot e^{st}\Big|_{s=0}=1 \qquad Done!!$$

THE END