5.4.4 Uniqueness of the Risk-Neutral Measure

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Outline:

1. Completeness of a market model

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2. Second fundamental theorem of asset pricing

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2. Second fundamental theorem of asset pricing
3. Lemma of Second fundamental theorem

Completeness of a market model

Theorem 5.4.8

A market model is complete if every derivative security can be hedged.

Suppose we have a market model with :

1. A filtration generated by a d-dimensional Brownian motion
2. A risk-neutral measure $\widetilde{P}$
   - We have solved the market price of risk equations

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t), \quad i = 1, \ldots, m$$

   - Using the resulting market prices of risk $\Theta_1(t), \ldots, \Theta_d(t)$ to define the Radon-Nikodym derivative process $Z(t)$.

2. A risk-neutral measure $\widetilde{P}$
   - have changed to the measure $\tilde{P}$ under which $\widetilde{W}(t)$ defined by

$$\widetilde{W}(t) = W(t) + \int_0^T \Theta(u)du$$

is a d-dimensional Brownian motion.

3. A $F(T)$-measurable random variable $V(T)$, which is the payoff of some derivative security.

- We would like to be sure we can hedge a short position in the derivative security whose payoff at time T is $V(T)$
- We can define V(t)

$$V(t) = \widetilde{E}\left[e^{-\int_t^T R(u)du} V(T) | F(t)\right]$$

so that D(t)V(t) satisfies

$$D(t)V(t) = \widetilde{E}\left[D(T)V(T)|F(t)\right]$$

and just as in

$$\widetilde{E}\left[D(t)V(t)|F(s)\right] = D(s)V(s)$$
$$0 \le s \le t \le T$$

- We see that $D(t)V(t)$ is a martingale under $\widetilde{P}$.
- According to the Martingale Representation , there are processes $\widetilde{\Gamma}_1(u), \ldots, \widetilde{\Gamma}_d(u)$ such that

$$D(t)V(t) = V(0) + \sum_{j=1}^{d} \int_0^T \widetilde{\Gamma}_j(u) d\widetilde{W}(u)$$

- Consider a portfolio value process that begins at $X(0)$. According to

$$d(D(t)X(t)) = \sum_{i=1}^{m} \Delta_i(t) d(D(t)S_i(t))$$

and

$$d(D(t)S_i(t)) = D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij} d\widetilde{W}_j(t)$$

- A portfolio value process would be

$$d(D(t)X(t)) = \sum_{i=1}^{m} \Delta_i(t) d(D(t)S_i(t))$$

$$= \sum_{j=1}^{d} \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

Or, equivalently,

$$D(t)X(t) = X(0) + \sum_{j=1}^{d} \int_0^T \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

- Comparing

$$D(t)V(t) = V(0) + \sum_{j=1}^{d} \int_0^T \widetilde{\Gamma}_j(u) d\widetilde{W}(u)$$

and

$$D(t)X(t) = X(0) + \sum_{j=1}^{d} \int_0^T \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

- To hedge the short position, we should take $X(0) = V(0)$ and choose the portfolio processes $\Delta_1(t), \dots, \Delta_m(t)$, so that the hedging equations

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

are satisfied.
- These are d equations in m unknown processes

$$\Delta_1(t), \dots, \Delta_m(t)$$

2nd fundamental theorem of asset pricing

Second fundamental theorem of asset pricing

Consider a market model that has a risk-neutral probability measure.

- The model is complete if and only if the risk-neutral probability measure is unique.

- Assume that the model is complete.
- We wish to show that there can be only one risk-neutral measure.

Model is complete $\Longrightarrow Q$ measure is unique

- Suppose the model has two risk-neutral measures, $\widetilde{P}_1$ and $\widetilde{P}_2$.
- Let A be a set in F, which we assumed at the beginning of this section is the same as $F(T)$.
- Consider the derivative security with payoff

$$V(T) = I_A \frac{1}{D(T)}$$

(Cont.) Model is complete $\Longrightarrow Q$ measure is unique

- Because the model is complete, a short position in this derivative security can be hedged. (there is a portfolio value process with some initial condition $X(0)$ that satisfies $X(T) = V(T)$.)
- Since both $\widetilde{P}_1$ and $\widetilde{P}_2$ are risk-neutral, the discounted portfolio value process $D(t)X(t)$ is a martingale under both $\widetilde{P}_1$ and $\widetilde{P}_2$.

(Cont.) Model is complete $\Longrightarrow$ Q measure is unique

- It follows that

$$\begin{aligned}\widetilde{P}_1(A) &= \widetilde{E}_1[D(T)V(T)] = \widetilde{E}_1[D(T)X(T)] \\ &= X(0) \\ &= \widetilde{E}_2[D(T)X(T)] = \widetilde{E}_2[D(T)V(T)] \\ &= \widetilde{P}_2(A)\end{aligned}$$

- Since A is an arbitrary set in F and $\widetilde{P}_1(A) = \widetilde{P}_2(A)$, these two risk-neutral measures are really the same.

(Cont.) Model is complete $\Longleftarrow Q$ measure is unique

- Suppose there is only one risk-neutral measure.
- This means first of all that the filtration for the model is generated by the d-dimensional Brownian motion driving the assets.

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- Uniqueness of the risk-neutral measure implies that the market price of risk equations

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t), \quad i = 1, \ldots, m$$

have only one solution $\Theta_1(t), \ldots, \Theta_d(t)$.

- For fixed t and w, these equations are of the form

$$Ax = b$$

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- For fixed t and w, these equations are of the form

$$Ax = b$$

Where A is the $m \times d$-dimensional matrix

$$A = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}(t) & \sigma_{m2}(t) & \cdots & \sigma_{md}(t) \end{bmatrix}$$

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- x is the d-dimensional column vector and b is the m-dimensional column vector

$$x = \begin{bmatrix} \Theta_1(t) \\ \Theta_2(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1(t) - R(t) \\ \alpha_2(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix}$$

- Our assumption that there is only one risk-neutral measure means that the system of $Ax = b$ has a unique solution x.

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- In other to be assures that every derivative security can be hedged, we must be able to solve the hedging equations $j = 1, \cdots, d$

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

for $\Delta_1(t), \cdots, \Delta_m(t)$no matter what values of $\frac{\widetilde{\Gamma}_j(t)}{D(t)}$ appear on the left-hand side.

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- For fixed t and w, the hedging equations are of the form

$$A^t y = c$$

Where A^t is the transpose of the matrix A, and y is the m-dimensional vector

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \Delta_1(t)S_1(t) \\ \Delta_2(t)S_2(t) \\ \vdots \\ \Delta_m(t)S_m(t) \end{bmatrix}$$

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- And c is the d-dimensional vector

$$c = \begin{bmatrix} \frac{\widetilde{\Gamma_1}(t)}{D(t)} \\ \frac{\widetilde{\Gamma_2}(t)}{D(t)} \\ \vdots \\ \frac{\widetilde{\Gamma_d}(t)}{D(t)} \end{bmatrix}$$

- In order to be assured that the market is complete, there must be a solution y to the system of $A^t y = c$, no matter what vector c appears on the right-hand side.

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- If there is always a solution $y_1, \cdots, y_m$, then there are portfolio processes $\Delta_i(t) = \frac{y_i}{S_i(t)}$ satisfying the hedging equations

$$\frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij} d\widetilde{W}_j(t)$$

  no matter what processes appear on the left-hand side of those equations.
- We could conclude that a short position in an arbitrary derivative security can be hedged.

(Cont.) Model is complete $\Longleftarrow$ Q measure is unique

- By the Lemma of Second fundamental theorem $\Rightarrow$ The uniqueness of the solution x to $Ax = b$ implies the existence of a solution y to $A^t y = c$.
- Consequently, uniqueness of the risk-neutral measure implies that the market model is complete.

Lemma of Second fundamental theorem

Lemma

Let A be an $m \times d$-dimensional matrix, b an m-dimensional vector, and c a d-dimensional vector. If the equation

$$Ax = b \tag{1}$$

has a unique solution x_0, then the equation

$$A^t y = c \tag{2}$$

has at least one solution y_0.

PROOF:

- We regard A as a mapping from $\mathbb{R}^d$ to $\mathbb{R}^m$ and define the $kernel$ of A to be

$$K(A) = \{x \in \mathbb{R}^d : Ax = 0\}$$

- If x_0 solves (1) and $x \in K(A)$ then $x_0 + x$ also solves (1).
- Thus, the assumption of a unique solution to (1) implies that $K(A)$ contains only the d-dimensional zero vector.

- The rank of A is defined to be the number of linearly independent columns of A.
- Because $K(A)$ contains only the d-dimensional zero vector, the rank of A must be d.
- Otherwise, we could find a linear combination of these columns that would be the m-dimensional zero vector, and the coefficients in this linear combination would give us a non-zero vector in $K(A)$.

Dimension Theorem

Let A is a linear transformation $A : V \to W$
If $dim(V) < \infty$
then $dim(kernel(A)) + rank(A) = dim(V)$

- By Dimension Theorem, $rank(A) = d$

$$dim(kernel(A)) + rank(A) = dim(V)$$
$$\Rightarrow 0 + rank(A) = d$$

Example: If rank of A is not d

- Let $A = \{a_1, \cdots, a_{d-1}, a_d\}$, where $\{a_i\}$ are the column vectors of A
- If the rank of A is $d - 1$ (It exists $\{w_i\}$ such that $a_d = \sum_{i=1}^{d-1} w_i a_i$)

(Cont.)Example: If rank of A is not d

- Then

$$
\begin{aligned}
Ax =& [a_1, \cdots, a_d] \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \\
=& [a_1 x_1 + a_2 x_2 + \cdots + a_d x_d] \\
=& \left[a_1 x_1 + a_2 x_2 + \cdots + x_d \sum_{i=1}^{d-1} w_i a_i \right] \\
=& [(x_1 + w_1 x_d) a_1 + (x_2 + w_2 x_d) a_2 \\
& + \cdots + (x_{d-1} - w_{d-1} x_d) a_d]
\end{aligned}
$$

(Cont.)Example: If rank of A is not d

- By solve above equation

$$s.t. \quad x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; or \; \begin{bmatrix} -w_1 c \\ -w_2 c \\ \vdots \\ c \end{bmatrix}, \quad \forall c \in \mathbb{R}$$

- Therefore, the dimension of $K(A)$ would not equal 0.

Theorem

Any matrix and its transpose have the same rank.

$$dim(R(A)) = dim(R(A^t))$$

- The rank of A^t is d as well.
- The rank of a matrix is also the dimension of its range space. The range space of A^t is

$$R(A^t) = \{z \in \mathbb{R}^d : z = A^t y, \quad \exists y \in \mathbb{R}^m\}$$

Theorem

Let W is a subspace of a vector space V, where $dim(V) < \infty$.
if $dim(W) = dim(V)$, then $W = V$.

- The dimension of $R(A^t)$ is also d and it is a subspace of $\mathbb{R}^d$, it must in fact equal to $\mathbb{R}^d$
- In other words, for every $z \in \mathbb{R}^d$, there is some $y \in \mathbb{R}^m$ such that $z = A^t y$.
- Hence (2) has at least one solution $y_0 \in \mathbb{R}^m$.

Thank you !