Kolmogrov Equation 介紹

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大綱

- 動機
- Feyman-Kac Equation
- Kolmogrov Equation
- 補充

動機

- 推導 Local Volatility Model 中的 Dupire’s Formula 時用到

Feyman-Kac Theorem

- 偏微分方程的解
- 折現的投組過程是Martingale ，故可以寫成期望折現值
- 兩個 f 的關聯是什麼（Feyman-Kac Thm）

Kolmogorov Equation

- 令 *X* 是一個隨機過程 ,其轉移密度函數為 *p*

$$dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u)$$

$$p(t, T; x, y)$$

- *SDE* 的解是Markov process (Shreve p.267)
- Kolmogorov Backward Equation
- Kolmogorov Forward Equation

Kolmogorov Backward Equation

- *t* , *x* 稱為 Backward Variable (Shreve p.291)

$$-p_t(t,T,x,y) = \beta(t,x)p_x(t,T,x,y) + \frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y). \qquad (6.9.43)$$

- 推導參考自Shreve p.291

1. Feyman-Kac Thm : 給定任意 *h(X(T))* , *g(t,x)* 滿足

$$g(t,x) = \mathbb{E}^{t,x} h\big(X(T)\big) = \int_0^\infty h(y)p(t,T,x,y)dy$$

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0.$$

2. 讓g對t,x偏微分

$$g_t(t,x) = \frac{\partial}{\partial t}\int_0^\infty h(y)p(t,T,x,y)dy = \int_0^\infty h(y)p_t(t,T,x,y)dy$$

$$g_x(t,x) = \int_0^\infty h(y)p_x(t,T,x,y)dy,$$

$$g_{xx}(t,x) = \int_0^\infty h(y)p_{xx}(t,T,x,y)dy.$$

3. 帶入原式可得

$$\int_0^\infty h(y)\left[p_t(t,T,x,y)+\beta(t,x)p_x(t,T,x,y)+\frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)\right]dy=0$$

4. 因為*h*是任取的，故等式成立

$$p_t(t,T,x,y)+\beta(t,x)p_x(t,T,x,y)+\frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)=0.$$

5. Kolmogorov Backward Equation 證明完畢

Kolmogorov Forward Equation

- *T* , *y* 稱為 Forward Variable (Shreve p.291)

$$\frac{\partial}{\partial T}p(t,T,x,y) = -\frac{\partial}{\partial y}\left(\beta(t,y)p(t,T,x,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T,y)p(t,T,x,y)\right). \quad (6.9.47)$$

- 推導參考自Shreve p.292

1. 對 h(X_T) 使用 Ito's Lemma

$$dh_b(X_u) = h_b'(X_u)dX_u + \tfrac{1}{2}h_b''(X_u)dX_u dX_u$$
$$= \left[h_b'(X_u)\beta(u,X_u) + \tfrac{1}{2}\gamma^2(u,X_u)h_b''(X_u)\right]du + h_b'(X_u)\gamma(u,X_u)dW_u$$

2. 對式子兩邊積分並取期望值

$$h_b(X_T) - h_b(X_t) = \int_t^T \left[h_b'(X_u)\beta(u,X_u) + \frac{1}{2}\gamma^2(u,X_u)h_b''(X_u)\right]du + \text{martingale part.}$$

$$\begin{aligned} E^{t,x}[h_b(X_T) - h_b(X_t)] &= \int_{-\infty}^{\infty} h_b(y)p(t,T,x,y)dy - h(x) \\ &= \int_t^T E^{t,x}[h_b'(X_u)\beta(u,X_u) + \frac{1}{2}\gamma^2(u,X_u)h_b''(X_u)]du \\ &= \int_t^T \int_{-\infty}^{\infty} \left[\underline{h_b'(y)\beta(u,y)} + \underline{\frac{1}{2}\gamma^2(u,y)h_b''(y)}\right] p(t,u,x,y)dydu. \end{aligned}$$

3. 對右式兩項分別使用 Integration By Part(因為 $h_b(x)$ 超過b 、小於0都是0)

$$\int_0^b \beta(u,y)p(t,u,x,y)h_b'(y)dy = h_b(y)\beta(u,y)p(t,u,x,y)|_0^b - \int_0^b h_b(y)\frac{\partial}{\partial y}(\beta(u,y)p(t,u,x,y))dy$$
$$= -\int_0^b h_b(y)\frac{\partial}{\partial y}(\beta(u,y)p(t,u,x,y))dy,$$

$$\int_0^b \gamma^2(u,y)p(t,u,x,y)h_b''(y)dy = -\int_0^b \frac{\partial}{\partial y}(\gamma^2(u,y)p(t,u,x,y))h_b'(y)dy$$
$$= \int_0^b \frac{\partial^2}{\partial y}(\gamma^2(u,y)p(t,u,x,y))h_b(y)dy$$

4. 帶入後原式中，並對 T 微分得

$$\int_0^b h_b(y)\frac{\partial}{\partial T}p(t,T,x,y)dy = -\int_0^b \frac{\partial}{\partial y}[\beta(T,y)p(t,T,x,y)]h_b(y)dy + \frac{1}{2}\int_0^b \frac{\partial^2}{\partial y^2}[\gamma^2(T,y)p(t,T,x,y)]h_b(y)dy$$

5. 移項整理後可得

$$\int_0^b h_b(y)\left[\frac{\partial}{\partial T}p(t,T,x,y)+\frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y))-\frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y))\right]dy=0.$$

6. 因為 $h_b(x)$ 是任取的所以有

$$\frac{\partial}{\partial T}p(t,T,x,y)+\frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y))-\frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y))=0.$$

7. Kolmogorov Forward Equation 證明完畢