CHAPTER 2.
Introduction to Stochastic Volatility Model

報告者：吳俊輝
指導教授：戴天時 教授

大綱

- 動機
- 隱含波動度的性質
- 原始模型、改進方法
  -Local Volatility Model （deterministic）
  -Stochastic Volatility Model （stochastic）
- 結論

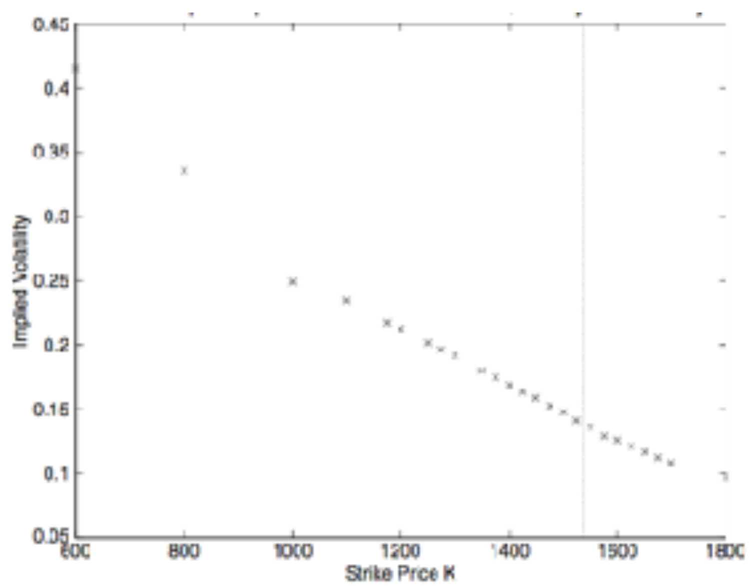
動機

- 隱含波動度的圖形與B-S公式預期的不同

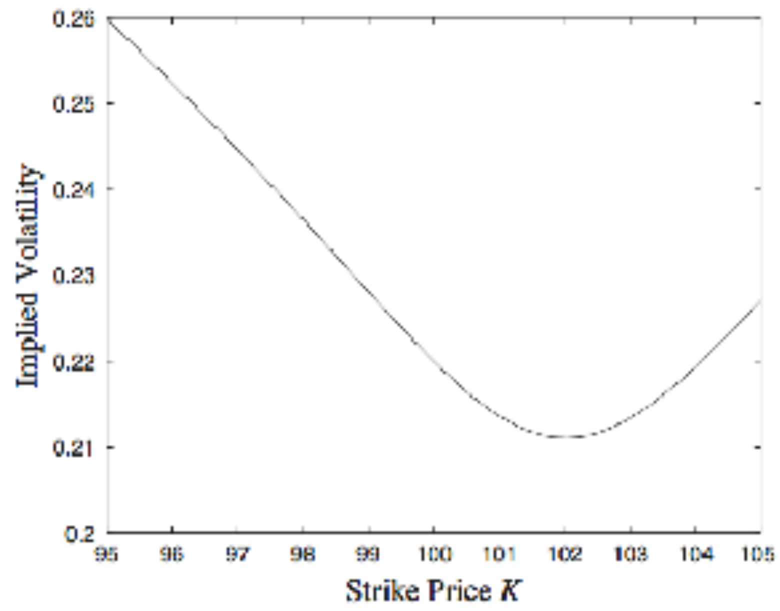
  - 隱含波動度：運用市場的報價反推B-S中的σ

  - 為何會有此現象：對崩盤有所恐懼，所以價外的put會付出較高的貼水

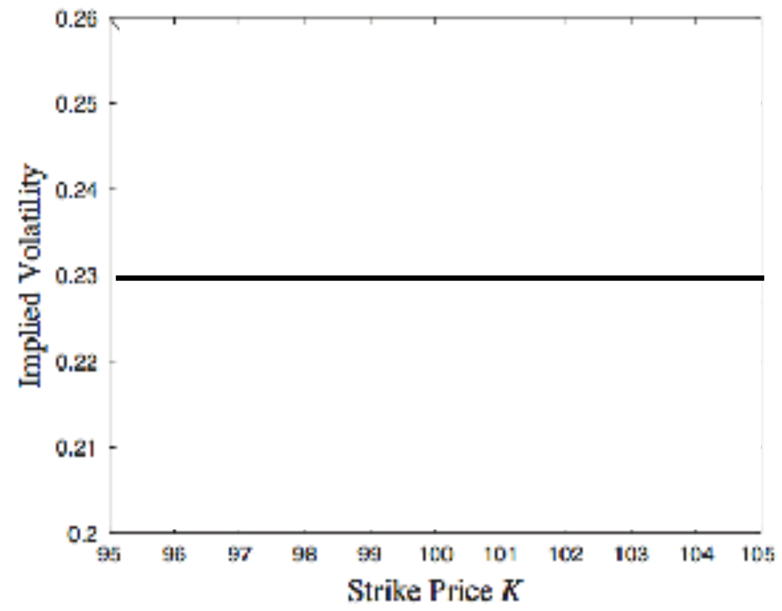
- 實證分析也說明股價報酬的波動度有隨機現象



市場看跌



漲跌都要付貼水



B-S的預期下

隱含波動度的性質

1. 可以將觀察到的報價寫成B-S的函數

$$C_{BS}(t,x;K,T;I) = C^{\text{obs}}. \tag{2.1}$$

2. 觀察到的報價一定會有下界

$$C^{\text{obs}} > C_{BS}(t,x;K,T;0)$$

because of the monotonicity of the Black–Scholes formula in the volatility parameter:

$$\frac{\partial C_{BS}}{\partial \sigma} = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}} > 0. \tag{2.2}$$

3. 買權跟賣權的隱含波動度會一樣（put-call parity）

4. 隱含波動度對履約價的斜率有上下界

$$\frac{\partial C^{obs}}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma}\frac{\partial I}{\partial K} \le 0,$$

履約價越高，賣權越貴、買權越便宜

$$\frac{\partial I}{\partial K} \le -\frac{\partial C_{BS}/\partial K}{\partial C_{BS}/\partial \sigma}.$$

$$\frac{\partial I}{\partial K} \ge -\frac{\partial P_{BS}/\partial K}{\partial P_{BS}/\partial \sigma}.$$

$$-\frac{\sqrt{2\pi}}{x\sqrt{T-t}}(1-N(d_2))e^{-r(T-t)+\frac{1}{2}d_1^2} \le \frac{\partial I}{\partial K} \le \frac{\sqrt{2\pi}}{x\sqrt{T-t}}N(d_2)e^{-r(T-t)+\frac{1}{2}d_1^2},$$

原始模型、修正方法

- B-S模型假設的價格過程：

  - 股價報酬率服從布朗運動： $dX_t = \mu X_t dt + \sigma X_t dW_t,$

  - 股價服從對數常態分佈　：$X_t = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right).$

- 修正方法：將 σ 條件放寬

  -Local Volatility Model （deterministic）

  -Stochastic Volatility Model （Stochastic）

Local Volatility Model

- 價格過程中sigma為標的物價格與時間的函數的模型

$$dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t,$$

- Time Dependent $\sigma(t,x) = \sigma(t),$

- Level Dependent $\sigma(t,x) = \kappa x^{-\gamma},$

- Non-Parametric (Dupire's Formula) 不預設函數形式

Time Dependent

- 價格過程：

$$dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t, \quad \sigma(t,x) = \sigma(t),$$

- 解開隨機偏微分方程得到：

$$X_T = X_t \exp\left(r(T-t) - \frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s) dW_s^\star \right),$$

- 如何解此隨機偏微分方程？*

- Xt 的分佈：

$$\log(X_T/X_t) \text{ is } \mathcal{N}\left(\left(r-\tfrac{1}{2}\overline{\sigma^2}\right)(T-t), \overline{\sigma^2}(T-t)\right).$$

$$\text{where } \overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(s)ds.$$

- 修正的B-S公式就是直接把 σ 換掉

- 假如觀察到隱含波動度有期間結構（但不是履約價函數），可以透過隱含波動度反找 σ 的積分

$$(T-t)I(t,T)^2 = \int_t^T \sigma^2(s)\,ds.$$

$$\int_{T_1}^{T_2} \sigma^2(s)\,ds = (T_2-t)I(t,T_2)^2 - (T_1-t)I(t,T_1)^2.$$

Level Dependent

- 價格過程：

$$dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t, \quad \sigma(t, x) = \kappa x^{-\gamma},$$

- 缺點：股價跟波動度會高度相關
- 要fit 陡峭的隱含波動度可以採用$3<\gamma<4$
- 書裡只介紹一種，其他沒有詳述

Dupire's Formula

1. 固定時間t,股價x ，選擇權價格可以寫成

$$C(T,K) = \mathbb{E}^{\star}\{e^{-r(T-t)}(X_T - K)^+ \mid X_t = x\}$$
$$= e^{-r(T-t)} \int_0^{\infty} (\xi - K)^+ p(t,x;T,\xi)\, d\xi,$$

2. 對K偏微分

$$\frac{\partial C}{\partial K}(T,K) = -e^{-r(T-t)} \int_0^{\infty} \mathbf{1}_{\{\xi > K\}} p(t,x;T,\xi)\, d\xi, \qquad (2.5)$$

$$\frac{\partial^2 C}{\partial K^2}(T,K) = e^{-r(T-t)} \int_0^{\infty} \delta(\xi - K) p(t,x;T,\xi)\, d\xi$$
$$= e^{-r(T-t)} p(t,x;T,K), \qquad (2.6)$$

3. 對T偏微分

$$\frac{\partial C}{\partial T}(T,K)$$
$$= e^{-r(T-t)} \int_0^\infty p(t,x;T,\xi) \mathscr{L}_T (\xi - K)^+ \, d\xi$$
$$- r e^{-r(T-t)} \int_0^\infty (\xi - K)^+ p(t,x;T,\xi) \, d\xi,$$
$$= e^{-r(T-t)} \int_0^\infty p(t,x;T,\xi) \left(\frac{1}{2} \sigma^2(T,\xi) \xi^2 \delta(\xi - K) + r\xi \mathbf{1}_{\{\xi > K\}} \right) d\xi$$
$$- r e^{-r(T-t)} \int_0^\infty (\xi - K) \mathbf{1}_{\{\xi > K\}} p(t,x;T,\xi) \, d\xi,$$

Kolmogorov Forward Equation (Shreve p.291) :

$$\mathscr{L}_T^* = \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \left(\sigma^2(T,\xi) \xi^2 \cdot \right) - \frac{\partial}{\partial \xi} (r \xi \cdot).$$

$$= \frac{1}{2}\sigma^2(T,K)K^2 e^{-r(T-t)} p(t,x;T,K)$$
$$+rKe^{-r(T-t)} \int_0^\infty \mathbf{1}_{\{\xi>K\}} p(t,x;T,\xi)\,d\xi$$
$$= \frac{1}{2}\sigma^2(T,K)K^2 \frac{\partial^2 C}{\partial K^2}(T,K) - rK\frac{\partial C}{\partial K}(T,K),$$

化簡算式：

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(T,K)K^2 \frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}, \quad T > t, \qquad (2.7)$$

整理　　：

$$\sigma^2(T,K) = \frac{\frac{\partial C}{\partial T}(T,K) + rK\frac{\partial C}{\partial K}(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}(T,K)}.$$

Stochastic Volatility Model

- 模型假設
- 股價密度函數比較
- 如何定價？
- Market Price of risk 選取、(K,T,t)問題

模型假設

- σ 為另一隨機過程, 無風險利率r為常數

$$
\begin{aligned}
dX_t &= \mu(Y_t)X_t\,dt + \sigma_t X_t\,dW_t^{(0)}, \\
\sigma_t &= f(Y_t), \\
dY_t &= \alpha(Y_t)\,dt + \beta(Y_t)\,dW_t^{(1)},
\end{aligned}
\tag{2.9}
$$

- Yt稱為Driving Factor(Driving Process)

  f 要是一個遞增、恆正的函數

- σ 寫成函數形式可以比較容易整合更多因子

- 常見的Driving Process

LN lognormal

$$dY_t = \alpha Y_t dt + \beta Y_t \, dW_t^{(1)},$$

OU Ornstein–Uhlenbeck

$$dY_t = \alpha(m - Y_t)dt + \beta \, dW_t^{(1)},$$

CIR Feller, square-root, or Cox–Ingersoll–Ross

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{Y_t}\, dW_t^{(1)}.$$

- OU , CIR Process 有mean reverting的特性（觀察drift term）

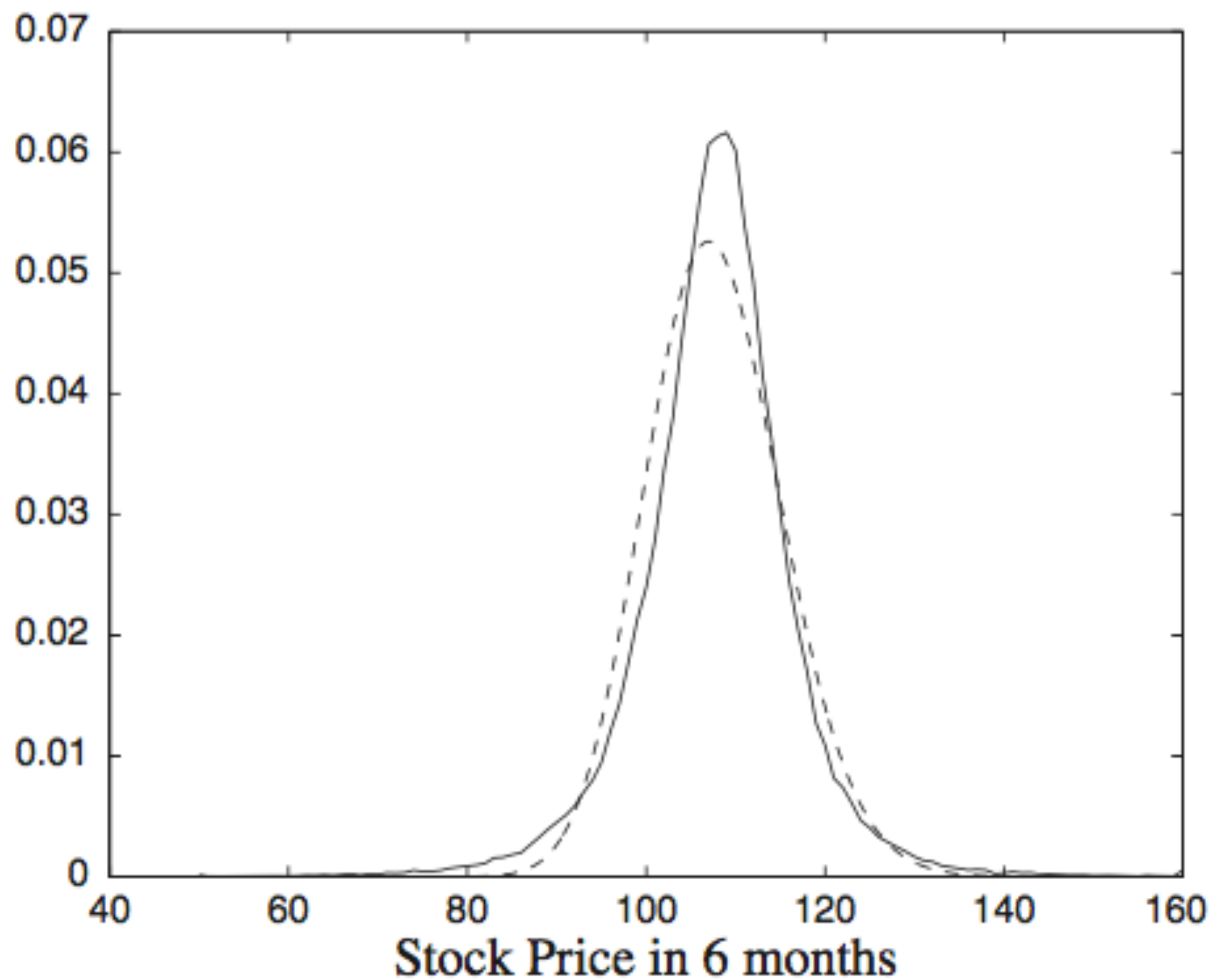
- 文獻中常用到的Stochastic Volatility Model

Authors	Correlation	$f(y)$	Y Process
Hull–White	$\rho = 0$	$f(y) = \sqrt{y}$	Lognormal
Scott	$\rho = 0$	$f(y) = e^y$	OU
Stein–Stein	$\rho = 0$	$f(y) = \lvert y \rvert$	OU
Ball–Roma	$\rho = 0$	$f(y) = \sqrt{y}$	CIR
Heston	$\rho \neq 0$	$f(y) = \sqrt{y}$	CIR

- 通常是因為容易分析，沒有太深的財務意涵

股價密度函數比較

- 實線是expOU模型下的股價密度函數（左端較厚）
  虛線是B-S模型下股價密度函數



如何定價？

- 回顧模型假設：

$$\begin{aligned} dX_t &= \mu(Y_t)X_t\,dt + \sigma_t X_t\,dW_t^{(0)}, \\ \sigma_t &= f(Y_t), \\ dY_t &= \alpha(Y_t)\,dt + \beta(Y_t)\,dW_t^{(1)}, \end{aligned} \tag{2.9}$$

- 將另一個布朗運動做分解：

$$W_t^{(1)} = \rho W_t^{(0)} + \sqrt{1-\rho^2}\,W_t^{\perp}. \tag{2.10}$$

- 造一個無風險的self financing投組（無風險投組賺無風險利率）

$$d\Pi_t = r\Pi_t\,dt$$

- 有兩個不確定性來源，因此無法只用股票避掉所有風險，因此需考慮另一個到期日較大的選擇權加入（市場不完備）

$$\Pi_t = N_t P^{(1)}(t, X_t, Y_t) - A_t X_t - \Sigma_t P^{(2)}(t, X_t, Y_t) \qquad (2.11)$$

- 因為投組符合 Self-Financing

$$d\Pi_t = N_t dP^{(1)}(t, X_t, Y_t) - A_t\, dX_t - \Sigma_t dP^{(2)}(t, X_t, Y_t),$$

- 接著對P1,P2 使用Ito's lemma

- 展開後

$$
\begin{aligned}
d\Pi_t = &\left(N_t\left[\frac{\partial}{\partial t}+\mathscr{L}_{(X,Y)}\right]P^{(1)} - A_t\mu(Y_t)X_t - \Sigma_t\left[\frac{\partial}{\partial t}+\mathscr{L}_{(X,Y)}\right]P^{(2)}\right)dt \\
&+\left(X_t f(Y_t)\left[N_t\frac{\partial P^{(1)}}{\partial x} - \Sigma_t\frac{\partial P^{(2)}}{\partial x} - A_t\right]\right. \\
&\left.+\rho\beta(Y_t)\left[N_t\frac{\partial P^{(1)}}{\partial y} - \Sigma_t\frac{\partial P^{(2)}}{\partial y}\right]\right)dW_t^{(0)} \\
&+\sqrt{1-\rho^2}\,\beta(Y_t)\left[N_t\frac{\partial P^{(1)}}{\partial y} - \Sigma_t\frac{\partial P^{(2)}}{\partial y}\right]dW_t^{\perp}.
\end{aligned}
$$

- 選取係數

$$
\Sigma_t = N_t\left(\frac{\partial P^{(2)}}{\partial y}\right)^{-1}\left(\frac{\partial P^{(1)}}{\partial y}\right), \qquad (2.13)
$$

$$
A_t = N_t\frac{\partial P^{(1)}}{\partial x} - \Sigma_t\frac{\partial P^{(2)}}{\partial x}. \qquad (2.14)
$$

- 不確定來源全部被消除，投組為無風險投組故可帶入

$$d\Pi_t = r\Pi_t\,dt,$$

- 帶入整理得到

$$\left(\frac{\partial P^{(1)}}{\partial y}\right)^{-1}\widehat{\mathscr{L}}P^{(1)} = \left(\frac{\partial P^{(2)}}{\partial y}\right)^{-1}\widehat{\mathscr{L}}P^{(2)}, \qquad (2.15)$$

$$\text{where}\quad \widehat{\mathscr{L}} = \frac{\partial}{\partial t} + \mathscr{L}_{(X,Y)} - (\mu(y) - r)x$$

- 觀察左右邊到期日不同，因此應該不是到期日的函數

令

$$\left(\frac{\partial P^{(1)}}{\partial y}\right)^{-1}\widehat{\mathscr{L}}P^{(1)} = \left(\frac{\partial P^{(2)}}{\partial y}\right)^{-1}\widehat{\mathscr{L}}P^{(2)} = -\alpha(y) + \beta(y)\Lambda(t,x,y),$$

$$\Lambda(t,x,y) = \rho \frac{(\mu(y)-r)}{f(y)} + \gamma(t,x,y)\sqrt{1-\rho^2}, \qquad (2.16)$$

- γ可以是任意函數（但是會關係到選取的風險中立測度）
  γ 又稱作market price of risk

- 整理完後得到微分方程（由2.15 推到 2.17）

$$\left(\frac{\partial P^{(1)}}{\partial y}\right)^{-1} \widehat{\mathscr{L}} P^{(1)} = \left(\frac{\partial P^{(2)}}{\partial y}\right)^{-1} \widehat{\mathscr{L}} P^{(2)}, \qquad (2.15)$$

$$\frac{\partial P}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2 P}{\partial x^2} + \rho \beta(y) x f(y) \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2(y) \frac{\partial^2 P}{\partial y^2} + r\left(x\frac{\partial P}{\partial x} - P\right) + (\alpha(y) - \beta(y)\Lambda(t,x,y))\frac{\partial P}{\partial y} = 0, \qquad (2.17)$$

風險中立測度評價

- 測度轉換 (Shreve)：

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, dP(\omega) \text{ for all } A \in \mathcal{F}. \qquad (5.2.1)$$

- 由Girsanov定理可將模型轉換，使得在新的測度之下投組H的折現過程是Martingale

$$V_t = \mathbb{E}^{\star(\gamma)}\{e^{-r(T-t)}H \mid \mathscr{F}_t\}. \qquad (2.21)$$

- Girsanov Theorem（Shreve P.212）：

Theorem 5.2.3 (Girsanov, one dimension). *Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\}, \tag{5.2.11}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du, \tag{5.2.12}$$

and assume that[1]

$$\mathbb{E}\int_0^T \Theta^2(u) Z^2(u)\,du < \infty. \tag{5.2.13}$$

Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by (5.2.1), the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

如何轉換？

- 將原本的價格過程分解，找尋適當的theta

$$W_t^{(0)\star} = W_t^{(0)} + \int_0^t \frac{(\mu(Y_s) - r)}{f(Y_s)} ds.$$

$$W_t^{\perp\star} = W_t^{\perp} + \int_0^t \gamma_s \, ds$$

- 有theta 之後可以造出新的測度

$$\frac{d\mathbb{P}^{\star(\gamma)}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T ((\theta_s^{(0)})^2 + (\theta_s^{\perp})^2) ds - \int_0^T \theta_s^{(0)} dW_s^{(0)} - \int_0^T \theta_s^{\perp} dW_s^{\perp}\right),$$

$$\theta_t^{(0)} = \frac{\mu(Y_t) - r}{f(Y_t)}, \qquad \theta_t^{\perp} = \gamma_t.$$

- 在新的機率測度 $\mathbb{P}^{\star(\gamma)}$下，$M_t = e^{-rt}V_t$是Martingale
  由Martingale Representation Theorem （Shreve）
  可以知道

$$M_t = M_0 + \int_0^t \eta_s^{(0)} dW_s^{(0)\star} + \int_0^t \eta_s^{\perp} dW_s^{\perp\star}, \qquad (2.22)$$

- 經過化簡，可以將投組化成：

$$dV_t = a_t dX_t + b_t r e^{rt} dt + c_t d\sigma_t,$$

- 說明了這是Incomplete Market Model

Market Price of risk 選取

- gamma（Market price of risk）可以任意選取，要如何選取？
- 本書採取的方法（p.72 ,L.5）是直接跟市場做連結，透過選擇權的價格直接推算出γ
- 估計模型參數時可以用MLE或是其他計量方法
  也可以使用Calibration（直接拿選擇權資料找最小MSE）

(K,T,t) 問題

- 如何透過選擇權價格、隱含波動度評價其他合約？

- 方法一：針對標的物做模型
  - K,T的問題都容易克服
  - t的問題最困難也最重要（模型的穩定度對路徑相關商品很重要）

- 方法二：針對隱含波動度做模型
  - 難以確認是否是無套利模型
  - 回推出來的標的物過程非馬可夫過程
  - 雖然價格fit的很好，但整個模型沒有推論能力（完全放棄t）