3.3 Brownian Motion

- $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$

- When $n \to \infty$ $\Rightarrow$ Brownian motion

Def. 3.3.1

- Let $(\Omega, \mathcal{F}, P)$ be a probability space
- For each $\omega \in \Omega$, there is a cont. function $\mathrm{W}(t), t \geq 0, W(0) = 0$ and that depends on ω.
- $\mathrm{W}(t), t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$\boldsymbol{W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_m) - W(t_{m-t})}$$

are independent and each of these is normally distributed with

$$\boldsymbol{E[W(t_{i+1}) - W(t_i)] = 0},$$
$$\boldsymbol{Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i}$$

Distribution of Brownian Motion

- $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed.
  1. $\boldsymbol{E[W(t_i)] = 0}$
  2. For $0 \le s < t$, $\boldsymbol{cov[W(s), W(t)] = s}$.

$$E[W(s), W(t)] = E\left[W(s)\big(W(t) - W(s)\big) + W^2(s)\right]$$
$$= E\left[W(s)]E[\big(W(t) - W(s)\big)] + E[W^2(s)\right]$$
$$= 0 + Var[W(s)] = s.$$

Covariance Matrix

- the covariance matrix for Brownian motion (i.e., for the m-dimensional random vector (W(t1), W(t2), ... , W(tm))) is

$$\begin{bmatrix} \mathrm{E}\left[W^2(t_1)\right] & \mathrm{E}\left[W(t_1)W(t_2)\right] & \cdots & \mathrm{E}\left[W(t_1)W(t_m)\right] \\ \mathrm{E}\left[W(t_2)W(t_1)\right] & \mathrm{E}\left[W^2(t_2)\right] & \cdots & \mathrm{E}\left[W(t_2)W(t_m)\right] \\ \vdots & \vdots & & \vdots \\ \mathrm{E}\left[W(t_m)W(t_1)\right] & \mathrm{E}\left[W(t_m)W(t_2)\right] & \cdots & \mathrm{E}\left[W^2(t_m)\right] \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

Prepare for m.g.f

- Recall the MGF of 1-dim normal r.v. $X \sim N(0, t)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2t}}, \varphi(u) = e^{\frac{1}{2}tu^2}$$

For Brownian motion

$$u_m W(t_m) + u_{m-1} W(t_{m-1}) + \cdots + u_1 W(t_1)$$
$$= u_m\big(W(t_m) - W(t_{m-1})\big) + (u_{m-1} + u_m)\big(W(t_{m-1}) - W(t_{m-2})\big)$$
$$+ \cdots + (u_1 + u_2 + \cdots + u_m) W(t_1)$$

moment-generating function

$$\varphi(u_1, u_2, \dots, u_m)$$
$$= Eexp\{u_m W(t_m) + u_{m-1} W(t_{m-1}) + \cdots + u_1 W(t_1)\}$$
$$= Eexp\{u_m\big(W(t_m) - W(t_{m-1})\big) + (u_{m-1} + u_m)\big(W(t_{m-1}) - W(t_{m-2})\big)$$
$$+ \cdots + (u_1 + u_2 + \cdots + u_m) W(t_1)\}$$
$$= Eexp\{u_m\big(W(t_m) - W(t_{m-1})\big)$$
$$* \mathrm{E}exp\big\{(u_{m-1} + u_m)\big(W(t_{m-1}) - W(t_{m-2})\big)\big\}$$
$$* \mathrm{E}exp\{(u_1 + u_2 + \cdots + u_m) W(t_1)\}$$
$$= exp\left\{\frac{1}{2}(u_m)^2 (t_m - t_{m-1})\right\} * exp\left\{\frac{1}{2}(u_{m-1} + u_m)^2 (t_{m-1} - t_{m-2})\right\} * \cdots$$
$$* exp\left\{\frac{1}{2}(u_1 + u_2 + \cdots + u_m)^2 (t_1)\right\}$$

Alternative characterizations of Brownian motion

- Let $(\Omega, \mathcal{F}, P)$ be a probability space

- For each $\omega \in \Omega$, there is a cont. function $\mathrm{W}(t), t \geq 0, W(0) = 0$ and that depends on ω. The following three properties are equivalent.

For all $0 = t_0 < t_1 < \cdots < t_m$

(1) the increments

$$\boldsymbol{W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-t})}$$

are independent and each of these is normally distributed with

$$\boldsymbol{E[W(t_{i+1}) - W(t_i)] = 0,}$$

$$\boldsymbol{Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i}$$

Alternative characterizations of Brownian motion

(2)The random variables $\boldsymbol{W(t_1), W(t_2), \ldots, W(t_m)}$ are jointly normally distributed with means equal to zero and covariance matrix.

$$\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

(3)The random variables $\boldsymbol{W(t_1), W(t_2), \ldots, W(t_m)}$ have the joint moment-generating function

$$exp\left\{\frac{1}{2}(u_m)^2(t_m - t_{m-1})\right\} * exp\left\{\frac{1}{2}(u_{m-1} + u_m)^2(t_{m-1} - t_{m-2})\right\} * \cdots * exp\left\{\frac{1}{2}(u_1 + u_2 + \cdots + u_m)^2(t_1)\right\}$$

Filtration for Brownian Motion

- Def. 3.3.3

Let $(\Omega, \mathcal{F}, P)$ be a prob. space defined on a Brownian motion $\mathrm{W}(t), t \geq 0$.

A filtration for the Brownian motion is a

collection of $\sigma - algebras\ \mathcal{F}(t)$, $\mathrm{t} \geq 0$, satisfying:

(1) **(Information accumulates)**

For $0 \leq s < t,\ \mathcal{F}(s) \subseteq \mathcal{F}(t)$

(2) **(Adaptivity)**

For each $t\ \geq 0,\ \mathrm{W}(t)\ is\ \mathcal{F}(t)\ msb$.

(3) **(Independence of future increments)**

For $0 \leq t < u$, the increment $\mathrm{W}(u) - \mathrm{W}(t)$ is indep. of $\mathcal{F}(t)$

Def. Adaptivity

- Let $\Delta\ (t),\ \ t \geq 0$, be a stochastic process. We say that $\Delta\ (t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geq 0$ the random variable $\Delta\ (t)$ is $\mathcal{F}(t)$ -measurable.

Martingale Property for Brownian Motion

- Theorem 3.3.4.

Brownian motion is a martingale.

proof:

Let $0 \leq s \leq t$ be given. Then

$$E[\mathrm{W}(t)|\mathcal{F}(s)] = E\big[\big(\mathrm{W}(t) - \mathrm{W}(s)\big) + \mathrm{W}(s)\big|\mathcal{F}(s)\big]$$

$$= E\big[\big(\mathrm{W}(t) - \mathrm{W}(s)\big)\big|\mathcal{F}(s)\big] + E[\mathrm{W}(s)|\mathcal{F}(s)]$$

$$= \mathrm{W}(s).$$

3.4 Quadratic Variation

- First-Order Variation

Choose a partition $\prod = \{t_0, t_1, \ldots, t_n\}$ of [0,T]

which is a set of times $0 = t_0 < t_1 < \cdots < t_n = T$

The maximum step size of the partition:

$$||\prod|| = \max_{j=0,\ldots,j-1} (t_{j+1} - t_j)$$

Then we define first-order variation:

$$FV_T(f) = \lim_{||\prod||\to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

Mean Value Theorem

- Let $f'(t)$ is defined everywhere

$\forall t_j, t_{j+1} \ \exists t_j^* \in \left[t_j, t_{j+1}\right]$ s.t.

$$\frac{f\left(t_{j+1}\right) - f(t_j)}{t_{j+1} - t_j} = f'(t_j^*)$$

Mean Value Theorem

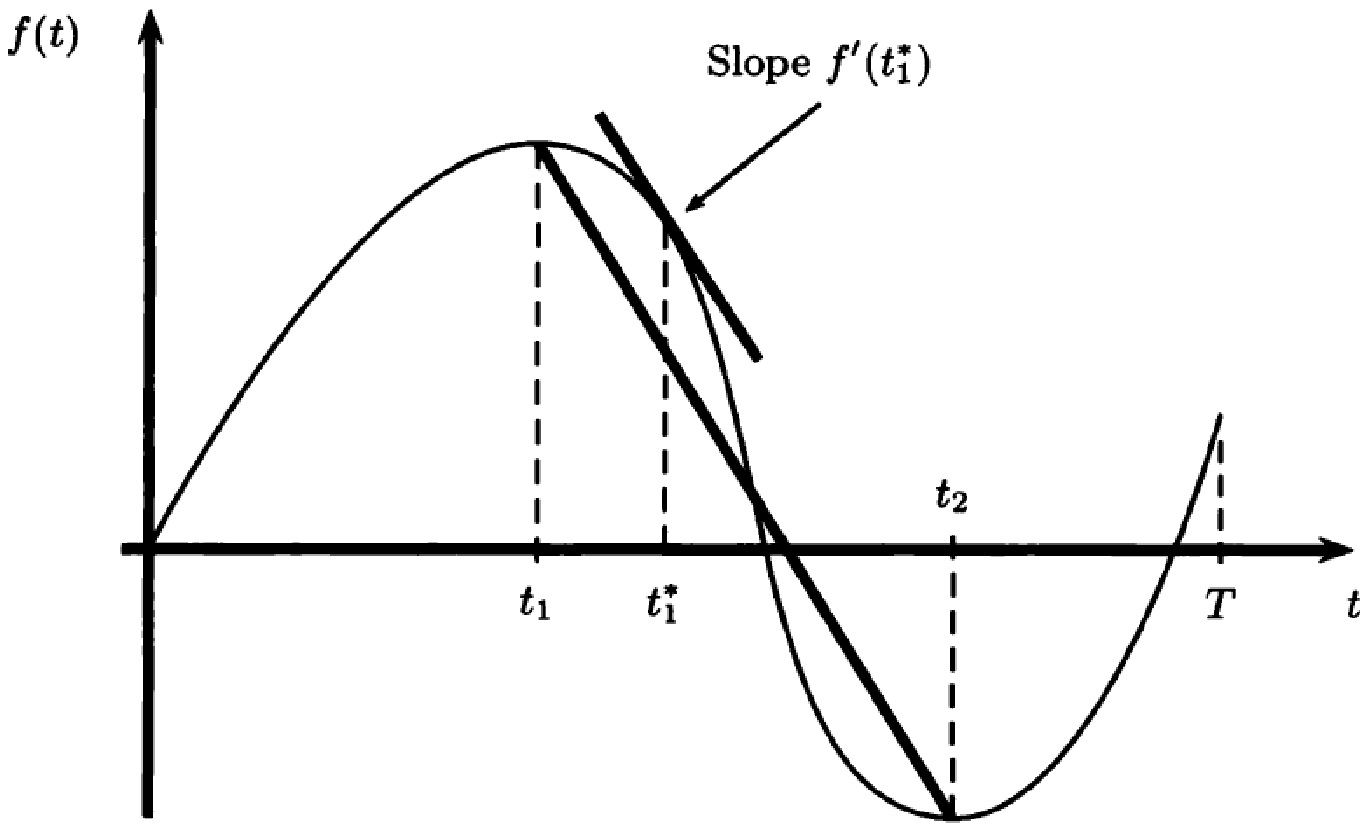

Fig. 3.4.2. Mean Value Theorem.

Mean Value Theorem

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = f'(t_j^*) \Longrightarrow f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$$

$$\Longrightarrow FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

$$= \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j) \quad \text{(Riemann sum)}$$

$$= \int_0^T |f'(t_j^*)|\, dt$$

Quadratic Variation

- Definition 3.4.1.
- Let $f(t)$ be a function defined for $0 \leq t \leq T$.

The quadratic variation of f up to time T is

$$[f,f](T) = \lim_{||\prod|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

where $\prod = \{t_0, t_1, \dots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$

Quadratic Variation

- Remark 3.4.2.
- Suppose the function f has a continuous derivative. By mean value theorem

$$\sum_{j=0}^{n-1}[f(t_{j+1}) - f(t_j)]^2 = \sum_{j=0}^{n-1}[f'(t_j^*)]^2(t_{j+1} - t_j)^2$$

$$\leq ||\Pi|| \sum_{j=0}^{n-1}[f'(t_j^*)]^2(t_{j+1} - t_j)$$

Quadratic Variation

and thus

$$[f,f](T) \le \lim_{||\prod||\to 0}[||\prod|| * \sum_{j=0}^{n-1}[f'(t_j^*)]^2(t_{j+1}-t_j)]$$

$$= \lim_{||\prod||\to 0}||\prod|| * \lim_{||\prod||\to 0}\sum_{j=0}^{n-1}[f'(t_j^*)]^2(t_{j+1}-t_j)$$

$$= \lim_{||\prod||\to 0}||\prod|| * \int_0^T [f'(t)]^2 dt = 0$$

In the last step of this argument, we use the fact that $f'(t)$ is continuous to ensure that $\int_0^T [f'(t)]^2 dt$ is finite.