CH4 : Stochastic Calculus
4.4 Ito-Doeblin Formula

Stochastic Calculus for Finance II
Continuous-Time Models

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Theorem 4.4.9 (Itô integral of a deterministic integrand). *Let* $W(s)$, $s \geq 0$, *be a Brownian motion, and let* $\Delta(s)$ *be a* nonrandom *function of time. Define* $I(t) = \int_0^t \Delta(s)\, dW(s)$. *For each* $t \geq 0$, *the random variable* $I(t)$ *is normally distributed with expected value zero and variance* $\int_0^t \Delta^2(s)\, ds$.

Theorem 4.4.9

Proof:

$$EI(t) = I(0) = 0$$

$$VarI(t) = EI^2(t) = \int_0^t \Delta^2(s)ds$$

moment-generating function:

$$\mathbb{E}e^{uI(t)} = \exp\left\{\frac{1}{2}u^2 \int_0^t \Delta^2(s)\,ds\right\} \text{ for all } u \in \mathbb{R}.$$

Because $\Delta(s)$ is not random, (4.4.30) is equivalent to

$$\mathbb{E}\exp\left\{uI(t) - \frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds\right\} = 1,$$

which may be rewritten as

$$\mathbb{E}\exp\left\{\int_0^t u\Delta(s)\,dW(s) - \frac{1}{2}\int_0^t \big(u\Delta(s)\big)^2\,ds\right\} = 1.$$

But the process

$$\exp\left\{\int_0^t u\Delta(s)\,dW(s) - \frac{1}{2}\int_0^t \left(u\Delta(s)\right)^2 ds\right\}$$

is a martingale.

$$E\exp\left\{\int_0^t u\Delta(s)dW(s) - \frac{1}{2}\int_0^t \left(u\Delta(s)\right)^2 ds\right\}$$
$$= \exp\left\{\int_0^0 u\Delta(s)dW(s) - \frac{1}{2}\int_0^0 \left(u\Delta(s)\right)^2 ds\right\} = 1$$

Example 4.4.10 (Vasicek interest rate model). Let $W(t)$, $t \geq 0$, be a Brownian motion. The Vasicek model for the interest rate process $R(t)$ is

$$dR(t) = \big(\alpha - \beta R(t)\big)\, dt + \sigma\, dW(t), \qquad (4.4.32)$$

where α, β, and σ are positive constants. Equation (4.4.32) is an example of a *stochastic differential equation.* It defines a random process, $R(t)$ in this case, by giving a formula for its differential, and the formula involves the random process itself and the differential of a Brownian motion.

The solution to the stochastic differential equation (4.4.32) can be determined in closed form and is

$$R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s}\, dW(s), \tag{4.4.33}$$

Proof

Let

$$f(t,x) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}x$$

and

$$X(t) = \int_0^t e^{\beta s}\, dW(s)$$

then

$$f_t(t,x) = -\beta e^{-\beta t}R(0) + \alpha e^{-\beta t} - \sigma\beta e^{-\beta t}x \; = \; \alpha - \beta f(t,x),$$

$$f_x(t,x) = \sigma e^{-\beta t},$$

$$f_{xx}(t,x) = 0.$$

$$
\begin{aligned}
& df\big(t, X(t)\big) \\
& = f_t\big(t, X(t)\big)\, dt + f_x\big(t, X(t)\big)\, dX(t) + \frac{1}{2} f_{xx}\big(t, X(t)\big)\, dX(t)\, dX(t) \\
& = \Big(\alpha - \beta f\big(t, X(t)\big)\Big)\, dt + \sigma\, dW(t).
\end{aligned}
$$

note:

Theorem 4.4.9 implies that R(t) is normally distributed

Properties with the Vasicek model

- the undesirable property that there is positive probability that $R(t)$ is negative
- the desirable property that the interest rate is mean-reverting

Example 4.4.11 (Cox-Ingersoll-Ross (CIR) interest rate model). Let $W(t)$, $t \geq 0$, be a Brownian motion. The Cox-Ingersoll-Ross model for the interest rate process $R(t)$ is

$$dR(t) = \big(\alpha - \beta R(t)\big)\, dt + \sigma \sqrt{R(t)}\, dW(t), \tag{4.4.34}$$

where α, β, and σ are positive constants. Unlike the Vasicek equation (4.4.32), the CIR equation (4.4.34) does not have a closed-form solution. The advantage of (4.4.34) over the Vasicek model is that the interest rate in the CIR model does not become negative.

expected value of R(t)

Set $f(t,x) = e^{\beta t} R(t)$

$$
\begin{aligned}
& d\big(e^{\beta t} R(t)\big) \\
& = df\big(t, R(t)\big) \\
& = f_t\big(t, R(t)\big)\, dt + f_x\big(t, R(t)\big)\, dR(t) + \frac{1}{2} f_{xx}\big(t, R(t)\big)\, dR(t)\, dR(t) \\
& = \beta e^{\beta t} R(t)\, dt + e^{\beta t}\big(\alpha - \beta R(t)\big)\, dt + e^{\beta t} \sigma \sqrt{R(t)}\, dW(t) \\
& = \alpha e^{\beta t}\, dt + \sigma e^{\beta t} \sqrt{R(t)}\, dW(t). \qquad (4.4.35)
\end{aligned}
$$

expected value of R(t)

Integration of both sides of (4.4.35) yields

$$\begin{aligned} e^{\beta t}R(t) &= R(0) + \alpha \int_0^t e^{\beta u}\,du + \sigma \int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u) \\ &= R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u). \end{aligned}$$

Recalling that the expectation of an Itô integral is zero, we obtain

$$e^{\beta t}\mathbb{E}R(t) = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)$$

or, equivalently,

$$\mathbb{E}R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right). \tag{4.4.36}$$

variance of R(t)

Set $X(t) = e^{\beta t} R(t)$

$$dX(t) = \alpha e^{\beta t}\, dt + \sigma e^{\beta t} \sqrt{R(t)}\, dW(t) = \alpha e^{\beta t}\, dt + \sigma e^{\frac{\beta t}{2}} \sqrt{X(t)}\, dW(t)$$

and $\mathbb{E}X(t) = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)$

variance of R(t)

According to the Ito-Doeblin formula

$f(x) = x^2$, $f'(x) = 2x$, and $f''(x) = 2$),

$$\begin{aligned} d\big(X^2(t)\big) &= 2X(t)\,dX(t) + dX(t)\,dX(t) \\ &= 2\alpha e^{\beta t} X(t)\,dt + 2\sigma e^{\frac{\beta t}{2}} X^{\frac{3}{2}}(t)\,dW(t) + \sigma^2 e^{\beta t} X(t)\,dt. \qquad (4.4.37) \end{aligned}$$

Integration of (4.4.37) yields

$$X^2(t) = X^2(0) + (2\alpha + \sigma^2) \int_0^t e^{\beta u} X(u)\,du + 2\sigma \int_0^t e^{\frac{\beta u}{2}} X^{\frac{3}{2}}(u)\,dW(u).$$

variance of R(t)

$$\begin{aligned}\mathbb{E}X^2(t) &= X^2(0) + \left(2\alpha + \sigma^2\right) \int_0^t e^{\beta u} \mathbb{E}X(u)\, du \\ &= R^2(0) + \left(2\alpha + \sigma^2\right) \int_0^t e^{\beta u} \Big(R(0) + \frac{\alpha}{\beta}(e^{\beta u} - 1) \Big)\, du \\ &= R^2(0) + \frac{2\alpha + \sigma^2}{\beta} \Big(R(0) - \frac{\alpha}{\beta} \Big) (e^{\beta t} - 1) + \frac{2\alpha + \sigma^2}{2\beta} \cdot \frac{\alpha}{\beta} (e^{2\beta t} - 1).\end{aligned}$$

variance of R(t)

$$
\begin{aligned}
\mathbb{E}R^2(t) &= e^{-2\beta t}\mathbb{E}X^2(t) \\
&= e^{-2\beta t}R^2(0) + \frac{2\alpha+\sigma^2}{\beta}\left(R(0)-\frac{\alpha}{\beta}\right)\left(e^{-\beta t}-e^{-2\beta t}\right) \\
&\qquad + \frac{\alpha\left(2\alpha+\sigma^2\right)}{2\beta^2}\left(1-e^{-2\beta t}\right).
\end{aligned}
$$

variance of R(t)

$$\begin{aligned}\operatorname{Var}\big(R(t)\big) &= \mathbb{E}R^2(t) - \big(\mathbb{E}R(t)\big)^2 \\ &= e^{-2\beta t}R^2(0) + \frac{2\alpha+\sigma^2}{\beta}\Big(R(0)-\frac{\alpha}{\beta}\Big)\big(e^{-\beta t}-e^{-2\beta t}\big) \\ &\qquad + \frac{\alpha\big(2\alpha+\sigma^2\big)}{2\beta^2}\big(1-e^{-2\beta t}\big) - e^{-2\beta t}R^2(0) \\ &\qquad - \frac{2\alpha}{\beta}R(0)\big(e^{-\beta t}-e^{-2\beta t}\big) - \frac{\alpha^2}{\beta^2}\big(1-e^{-\beta t}\big)^2 \\ &= \frac{\sigma^2}{\beta}R(0)\Big(e^{-\beta t}-e^{-2\beta t}\Big) + \frac{\alpha\sigma^2}{2\beta^2}\Big(1-2e^{-\beta t}+e^{-2\beta t}\Big).\end{aligned} \tag{4.4.38}$$

In particular,

$$\lim_{t\to\infty}\operatorname{Var}\big(R(t)\big) = \frac{\alpha\sigma^2}{2\beta^2}$$