CH4 : Stochastic Calculus
4.4 Ito-Doeblin Formula

Stochastic Calculus for Finance II
Continuous-Time Models

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4.4.1 Formula for Brownian Motion

- **chain rule** from ordinary calculus

$$df(W(t)) = f'(W(t))\,W'(t)\,dt = f'(W(t))\,dW(t)$$

4.4.1 Formula for Brownian Motion

- But W has **nonzero quadratic variation**

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2}f''(W(t))\,dt$$

The Ito-Doeblin formula in **differential form**.

4.4.1 Formula for Brownian Motion

- The Ito-Doeblin formula in integral form:

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))\, dW(u) + \frac{1}{2}\int_0^t f''(W(u))\, du$$

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$f\big(T, W(T)\big) = f\big(0, W(0)\big) + \int_0^T f_t\big(t, W(t)\big)\,dt + \int_0^T f_x\big(t, W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t, W(t)\big)\,dt. \quad (4.4.3)$$

Theorem 4.4.1 (Ito-Doeblin formula for Brownian motion)

- Proof:

Fix $T > 0$, and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of [0,T]

1.written as the sum of the changes

2.use Taylor’s formula

Preview:

$$
\begin{aligned}
& f(t_{j+1}, x_{j+1}) - f(t_j, x_j) \\
& = f_t(t_j, x_j)(t_{j+1} - t_j) + f_x(t_j, x_j)(x_{j+1} - x_j) \\
& \qquad + \frac{1}{2} f_{xx}(t_j, x_j)(x_{j+1} - x_j)^2 + f_{tx}(t_j, x_j)(t_{j+1} - t_j)(x_{j+1} - x_j) \\
& \qquad + \frac{1}{2} f_{tt}(t_j, x_j)(t_{j+1} - t_j)^2 + \text{ higher-order terms.}
\end{aligned}
\tag{4.4.8}
$$

replace X_j by $W(t_j)$, X_{j+1} by $W(t_{j+1})$, and sum:

$$
\begin{aligned}
&f(T, W(T)) - f(0, W(0)) \\
&= \sum_{j=0}^{n-1} \left[f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j)) \right] \\
&= \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x(t_j, W(t_j))\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{xx}(t_j, W(t_j))\big(W(t_{j+1}) - W(t_j)\big)^2 \\
&\quad + \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)\big(W(t_{j+1}) - W(t_j)\big) \\
&\quad + \frac{1}{2} \sum_{j=0}^{n-1} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 + \text{ higher-order terms.} \qquad (4.4.9)
\end{aligned}
$$

take the limit as $\|\Pi\| \to 0$,

- the left-hand side of (4.4.9) is unaffected
- The first term on the right-hand side of (4.4.9):

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) = \int_0^T f_t\big(t, W(t)\big)dt$$

- the second term:

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) = \int_0^T f_x\big(t, W(t)\big)dW(t)$$

- the third term:

$$\lim_{\|\Pi\|\to 0} \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 = \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

- the fourth term

$$
\begin{aligned}
&\lim_{\|\Pi\|\to 0} \left| \sum_{j=0}^{n-1} f_{tx}\big(t_j, W(t_j)\big)(t_{j+1} - t_j)\big(W(t_{j+1}) - W(t_j)\big) \right| \\
&\leq \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left| f_{tx}\big(t_j, W(t_j)\big)\right| \cdot (t_{j+1} - t_j) \cdot \left| W(t_{j+1}) - W(t_j) \right| \\
&\leq \lim_{\|\Pi\|\to 0} \max_{0\leq k\leq n-1} \left| W(t_{k+1}) - W(t_k) \right| \cdot \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left| f_{tx}\big(t_j, W(t_j)\big)\right| (t_{j+1} - t_j) \\
&= 0 \cdot \int_0^T \left| f_{tx}\big(t, W(t)\big) \, dt \;=\; 0. \qquad (4.4.10)
\end{aligned}
$$

- the fifth term

$$\begin{aligned}
&\lim_{\|\Pi\|\to 0}\left|\frac{1}{2}\sum_{j=0}^{n-1} f_{tt}\big(t_j, W(t_j)\big)(t_{j+1}-t_j)^2\right| \\
&\quad \le \lim_{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=0}^{n-1}\left|f_{tt}\big(t_j, W(t_j)\big)\right|\cdot(t_{j+1}-t_j)^2 \\
&\quad \le \frac{1}{2}\lim_{\|\Pi\|\to 0}\max_{0\le k\le n-1}(t_{k+1}-t_k)\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left|f_{tt}\big(t_j, W(t_j)\big)\right|(t_{j+1}-t_j) \\
&\quad = \frac{1}{2}\cdot 0\cdot\int_0^T f_{tt}\big(t, W(t)\big)\,dt \;=\; 0. \qquad (4.4.11)
\end{aligned}$$

Remark 4.4.2.

- $(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j))$ has limit zero

$\Longrightarrow dtdW(t) = 0$

- $(t_{j+1} - t_j)(t_{j+1} - t_j)$has limit zero

$\Longrightarrow dtdt = 0$

Remark 4.4.2.

- In differential form it becomes

$$
\begin{aligned}
&df\big(t, W(t)\big) \\
&= f_t\big(t, W(t)\big)\,dt + f_x\big(t, W(t)\big)\,dW(t) + \frac{1}{2} f_{xx}\big(t, W(t)\big)\,dW(t)\,dW(t) \\
&\qquad + f_{tx}\big(t, W(t)\big)\,dt\,dW(t) + \frac{1}{2} f_{xx}\big(t, W(t)\big)\,dt\,dt,
\end{aligned}
$$

but

$$
dW(t)\,dW(t) = dt, \quad dt\,dW(t) = dW(t)\,dt = 0, \quad dt\,dt = 0, \qquad (4.4.12)
$$

Remark 4.4.2.

- The Ito-Doeblin formula in differential form simplifies to

$$df\big(t,W(t)\big) = f_t\big(t,W(t)\big)\,dt + f_x\big(t,W(t)\big)\,dW(t) + \frac{1}{2}f_{xx}\big(t,W(t)\big)\,dt. \quad (4.4.13)$$

Figure 4.4.1

- The first-order approximation has an error due to the convexity of the function f(x).

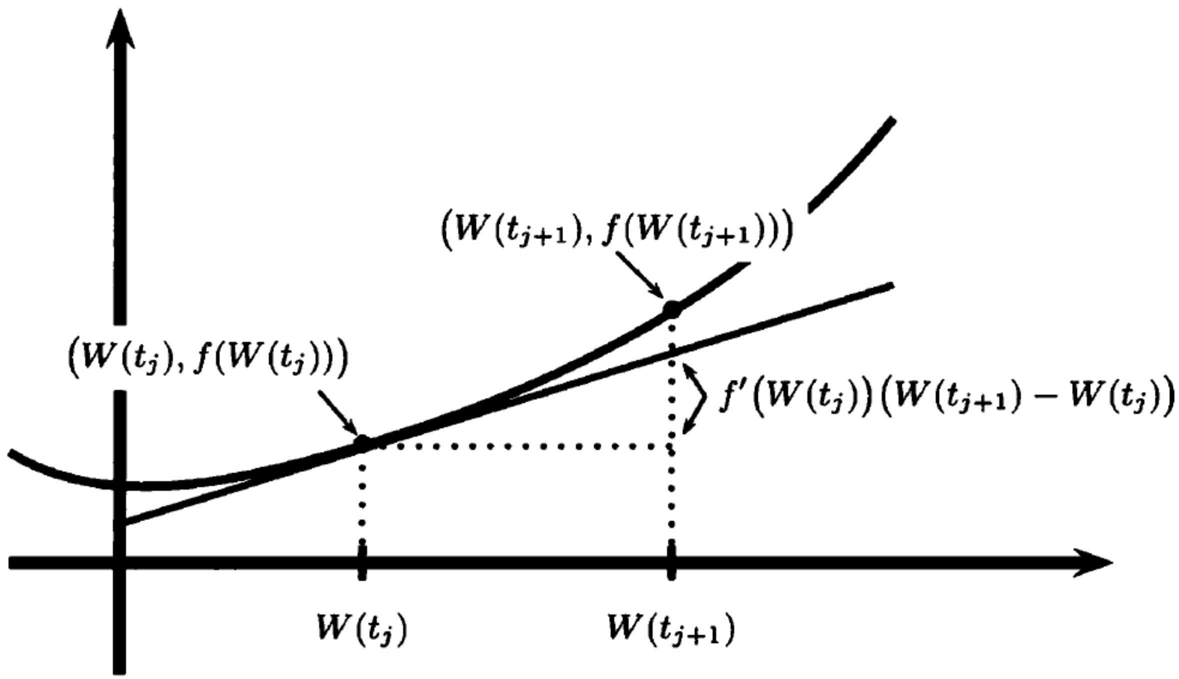

Fig. 4.4.1. Taylor approximation to $f(W(t_{j+1})) - f(W(t_j))$.

In other words,

$$f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) = f'\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big) + \text{ small error,} \tag{4.4.14}$$

and

$$\begin{aligned} f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) = f'\big(W(t_j)\big)&\big(W(t_{j+1}) - W(t_j)\big) \\ &+\frac{1}{2}f''\big(W(t_j)\big)\big(W(t_{j+1}) - W(t_j)\big)^2 \\ &+ \text{smaller error.} \end{aligned} \tag{4.4.15}$$

For example, with $f(x) = \frac{1}{2}x^2$, this formula says that

$$
\begin{aligned}
\frac{1}{2}W^2(T) &= f(W(T)) - f(W(0)) \\
&= \int_0^T f'(W(t))\,dW(t) + \frac{1}{2}\int_0^t f''(W(t))\,dt \\
&= \int_0^T W(t)\,dW(t) + \frac{1}{2}T.
\end{aligned}
$$

- Rearranging terms, we have formula (4.3.6)

4.4.2 Formula for Ito Processes

Definition 4.4.3. *Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration. An* Itô process *is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)\, dW(u) + \int_0^t \Theta(u)\, du, \tag{4.4.16}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes.[2]

4.4.2 Formula for Ito Processes

Lemma 4.4.4. *The quadratic variation of the Itô process (4.4.16) is*

$$[X,X](t) = \int_0^t \Delta^2(u)\,du. \tag{4.4.17}$$

Proof:

Set $I(t) = \int_0^t \Delta(u)dW(u)$, $R(t) = \int_0^t \Theta(u)du$

$\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of [0,t]

quadratic variation:

$$\sum_{j=0}^{n-1} \left[X(t_{j+1}) - X(t_j)\right]^2 = \sum_{j=0}^{n-1} \left[I(t_{j+1}) - I(t_j)\right]^2 + \sum_{j=0}^{n-1} \left[R(t_{j+1}) - R(t_j)\right]^2$$

$$+2 \sum_{j=0}^{n-1} \left[I(t_{j+1}) - I(t_j)\right]\left[R(t_{j+1}) - R(t_j)\right].$$

AS $\|\Pi\| \to 0$

- according to Theorem 4.3.1(vi)

the first term

$$[I, I](t) = \int_0^t \Delta^2(u)du$$

The absolute value of the second term is bounded above by

$$\max_{0\le k\le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|R(t_{j+1}) - R(t_j)\right|$$

$$= \max_{0\le k\le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|\int_{t_j}^{t_{j+1}} \Theta(u)\,du\right|$$

$$\le \max_{0\le k\le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\Theta(u)|\,du$$

$$= \max_{0\le k\le n-1} \left|R(t_{k+1}) - R(t_k)\right| \cdot \int_0^t |\Theta(u)|\,du,$$

The absolute value of the third term is bounded above by

$$2 \max_{0 \le k \le n-1} \left| I(t_{k+1}) - I(t_k) \right| \cdot \sum_{j=0}^{n-1} \left| R(t_{j+1}) - R(t_j) \right|$$

$$\le 2 \max_{0 \le k \le n-1} \left| I(t_{k+1}) - I(t_k) \right| \cdot \int_0^t |\Theta(u)| \, du,$$

- AS $||\Pi|| \to 0$, the second term and the third term has limit 0 because $I(t)$ and $R(t)$ is continuous

$$\Rightarrow [X,X](t) = [I,I](t) = \int_0^t \Delta^2(u)du.$$

Lemma 4.4.4 in differential form

$$dX(t) = \Delta(t)\, dW(t) + \Theta(t)\, dt$$

$$\begin{aligned} dX(t)\, dX(t) &= \Delta^2(t)\, dW(t)\, dW(t) + 2\Delta(t)\Theta(t)\, dW(t)\, dt + \Theta^2(t)\, dt\, dt \\ &= \Delta^2(t)\, dt. \end{aligned} \tag{4.4.19}$$

Definition 4.4.5. *Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $\Gamma(t)$, $t \geq 0$, be an adapted process. We define the* integral with respect to an Itô process[3]

$$\int_0^t \Gamma(u)\,dX(u) = \int_0^t \Gamma(u)\Delta(u)\,dW(u) + \int_0^t \Gamma(u)\Theta(u)\,du. \tag{4.4.20}$$

Theorem 4.4.6 (Itô-Doeblin formula for an Itô process). *Let* $X(t)$, $t \geq 0$, *be an Itô process as described in Definition 4.4.3, and let* $f(t,x)$ *be a function for which the partial derivatives* $f_t(t,x)$, $f_x(t,x)$, *and* $f_{xx}(t,x)$ *are defined and continuous. Then, for every* $T \geq 0$,

$$
\begin{aligned}
& f\big(T, X(T)\big) \\
&= f\big(0, X(0)\big) + \int_0^T f_t\big(t, X(t)\big)\, dt + \int_0^T f_x\big(t, X(t)\big)\, dX(t) \\
&\qquad + \frac{1}{2} \int_0^T f_{xx}\big(t, X(t)\big)\, d[X, X](t) \\
&= f\big(0, X(0)\big) + \int_0^T f_t\big(t, X(t)\big)\, dt + \int_0^T f_x\big(t, X(t)\big) \Delta(t)\, dW(t) \\
&\qquad + \int_0^T f_x\big(t, X(t)\big) \Theta(t)\, dt + \frac{1}{2} \int_0^T f_{xx}\big(t, X(t)\big) \Delta^2(t)\, dt. \qquad (4.4.22)
\end{aligned}
$$

Theorem 4.4.6

- Proof:

Fix $T > 0$, and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of [0,T]

1.written as the sum of the changes

2.use Taylor's formula

$$
\begin{aligned}
&f\big(T, X(T)\big) - f\big(0, X(0)\big) \\
&= \sum_{j=0}^{n-1} f_t\big(t_j, X(t_j)\big)(t_{j+1} - t_j) + \sum_{j=0}^{n-1} f_x\big(t_j, X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big) \\
&\qquad + \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}\big(t_j, X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big)^2 \\
&\qquad + \sum_{j=0}^{n-1} f_{tx}\big(t_j, X(t_j)\big)(t_{j+1} - t_j)\big(X(t_{j+1}) - X(t_j)\big) \\
&\qquad + \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}\big(t_j, X(t_j)\big)(t_{j+1} - t_j)^2 + \text{ higher-order terms.} \qquad (4.4.21)
\end{aligned}
$$

take the limit as $\|\Pi\| \to 0$,

- The first term on the right-hand side of (4.4.21):

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f_t(t_j, X(t_j))(t_{j+1} - t_j) = \int_0^T f_t\big(t, X(t)\big)dt$$

- the second term:

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} f_x(t_j, X(t_j))(X(t_{j+1}) - X(t_j)) = \int_0^T f_x\big(t, X(t)\big)dX(t)$$

$$= \int_0^T f_x\big(t, X(t)\big)\Delta(t)dW(t) + \int_0^T f_x\big(t, X(t)\big)\Theta(t)dt$$

- the third term:

$$\lim_{\|\Pi\|\to 0} \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j, X(t_j))(X(t_{j+1}) - X(t_j))^2$$

$$= \frac{1}{2}\int_0^T f_{xx}\big(t, X(t)\big)d[X,X](t) = \frac{1}{2}\int_0^T f_{xx}\big(t, X(t)\big)d[X,X](t)$$

$$= \frac{1}{2}\int_0^T f_{xx}\big(t, X(t)\big)\Delta^2(t)dt$$

- The last two sums on the right-hand side have zero limits as $\|\Pi\| \to 0$.

Remαrk 4.4.7 (Summary of stochastic calculus)

- In differential notation

$$df\big(t, X(t)\big) = f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t). \tag{4.4.23}$$

$$\begin{aligned} df\big(t, X(t)\big) = {} & f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\Delta(t)\,dW(t) \\ & + f_x\big(t, X(t)\big)\Theta(t)\,dt + \frac{1}{2}f_{xx}\big(t, X(t)\big)\Delta^2(t)\,dt. \end{aligned} \tag{4.4.24}$$

4.4.3 Examples

Example 4.4.8 (Generalized geometric Brownian motion). Let $W(t)$, $t \geq 0$, be a Brownian motion, let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds. \tag{4.4.25}$$

Example4.4.8

- Then

$$dX(t) = \sigma(t)dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt$$

- Consider an asset price process given by

$$S(t) = S(0)e^{X(t)} = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}$$

where S(0) is nonrandom and positive.

Example4.4.8

- We may write

$$S(t) = f\big(X(t)\big), f(x) = S(0)e^{x}$$

$$f'(x) = S(0)e^{x},\ f''(x) = S(0)e^{x}$$

Example4.4.8

- According to the Ito-Doeblin formula

$$\begin{aligned} dS(t) &= df\big(X(t)\big) \\ &= f'\big(X(t)\big)\,dX(t) + \frac{1}{2}f''\big(X(t)\big)\,dX(t)\,dX(t) \\ &= S(0)e^{X(t)}\,dX(t) + \frac{1}{2}S(0)e^{X(t)}\,dX(t)\,dX(t) \\ &= S(t)\,dX(t) + \frac{1}{2}S(t)\,dX(t)\,dX(t) \\ &= \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t). \end{aligned}$$