Game options

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Abstract

- Player:

seller A, buyer B

Abstract

- Both buyer and seller to stop them at any time

then the buyer can exercise the right

a specified security (S_t) for certain agreed price (K)

Abstract

- If the contract is terminated by A

then A must pay certain penalty to B

Abstract

- Analysis:

game contingent claims (GCC)

optimal stopping games (Dynkin's games)

Abstract

- Characteristic:

cheaper than usual American options

diversify financial markets

Introduction

- What is the fair price V^* for such contract?

Introduction

- Based on hedging

V^* should be the minimal capital

invest it into a self-financing portfolio

cover liability

Introduction

- A standard (B_t, S_t)-securities market

nonrandom (riskless) component B_t

random (risky) component S_t

Introduction

- A probability space $(\Omega, \mathcal{F}, P)$

with a stochastic process $S_t \geq 0$

σ-algebras $\mathcal{F}_t \subset \mathcal{F}$,

$\mathcal{F}_t$ is generated by all $S_u, 0 \leq u \leq t$

Introduction

- Two right continuous with left limits stochastic payoff processes:

$\mathrm{X}_t \geq \mathrm{Y}_t \geq 0$, adapted to the filtration $\mathcal{F}_t$

Introduction

- A game contingent claim (GCC):

B exercises => payoff = Y_t
A cancels => payoff = X_t
same time => payoff = Y_t

$$\delta_t = X_t - Y_t \geq 0$$

Introduction

Assuming that clairvoyance is not possible A and B have to use only stopping times with respect to the filtration $\{\mathscr{F}_t\}$ as their cancellation and exercise times.

Introduction

- European options:

$Y_t = 0$ for $t < T$ and $Y_t = Y_T \geq 0$ for $t = T$

- American options:

if the penalty is chosen large enough

Introduction

- Discrete time: CRR-model
- continuous time: geometric Brownian motion
- Markov case: $Y_t = \beta^t Y(S_t), X_t = \beta^t X(S_t), \beta \leq 1$

Discrete time

- Based on CRR-model
- $\Omega = \{1, -1\}^N$
- $\omega = (\omega_1, \omega_2, \ldots, \omega_N), \omega_i = 1 \; or - 1$

Discrete time

- with the product probability:

$$P = \{p, q\}^N, q = 1 - p, 0 < p < 1$$

- $p(\omega) = p^k q^{N-k}, k = \frac{1}{2}(N + \sum_{i=1}^{N} \omega_i)$

Discrete time

- savings account:

$$B_n = (1+r)^{\,n} B_0, B_0 > 0, r > 0$$

- stock price:

$$S_n = S_0 \prod_{k=1}^{N} (1+\rho_k), S_0 > 0$$

where $\rho_k(\omega) = \frac{1}{2}\Big(a + b + \omega_k\big(b - a\big)\Big),\ -1 < a < r < b$

Discrete time

- portfolio strategy π:

$$Z_0^{\pi} = z > 0, \pi = (\pi_1, \pi_2, \dots, \pi_N)$$

- at time n:

$$\pi_n = (\beta_n, \gamma_n), Z_n^{\pi} = \beta_n B_n + \gamma_n S_n$$

- self-financing:

$$B_{n-1}(\beta_n - \beta_{n-1}) + S_{n-1}(\gamma_n - \gamma_{n-1}) = 0$$

Discrete time

- cancellation time:

$$\sigma, \tau \in \mathcal{F}_{0N}$$

$$\sigma \wedge \tau \stackrel{\text{def}}{=} \min(\sigma, \tau)$$

- payoff:

$$\mathrm{R}(\sigma, \tau) \stackrel{\text{def}}{=} X_\sigma \mathrm{I}_{\sigma < \tau} + Y_\tau \mathrm{I}_{\tau \leq \sigma}$$

Discrete time

A hedge against a GCC with a maturity date N is a pair (σ, π) of a stopping time $\sigma \in \mathcal{T}_{0N}$ and a self-financing portfolio strategy π such that $Z^{\pi}_{\sigma \wedge n} \geq R(\sigma, n)$ for all $n = 0, 1, \ldots, N$.

The fair price V^* of a GCC is the infimum of $V \geq 0$ such that there exists a hedge (σ, π) against this GCC with $Z^{\pi}_0 = V$.

Theorem 2.1

Theorem 2.1 *Let* $P^* = \{p^*, 1 - p^*\}^N$ *be the probability on the space* Ω *with* $p^* = \frac{r-a}{b-a}$, $N < \infty$ *and* E^* *denotes the corresponding expectation. Then the fair price* V^* *of the above GCC equals* V^*_{0N} *which can be obtained from the recursive relations* $V^*_{NN} = (1+r)^{-N} Y_N$ *and for* $n = 0, 1, \ldots, N-1$

$$V^*_{nN} = \min((1+r)^{-n} X_n, \max((1+r)^{-n} Y_n, E^*(V^*_{n+1N} | \mathcal{F}_n))). \qquad (2.7)$$

Theorem 2.1

Moreover, for $n = 0, 1, \ldots, N$,

$$V^*_{nN} = \min_{\sigma \in \mathcal{J}_{nN}} \max_{\tau \in \mathcal{J}_{nN}} E^* \left((1+r)^{-\sigma \wedge \tau} R(\sigma, \tau) \middle| \mathcal{F}_n \right) \tag{2.8}$$

$$= \max_{\tau \in \mathcal{J}_{nN}} \min_{\sigma \in \mathcal{J}_{nN}} E^* \left((1+r)^{-\sigma \wedge \tau} R(\sigma, \tau) \middle| \mathcal{F}_n \right).$$

(Back)

(Back1)

Theorem 2.1

Furthermore, for each $n = 0, 1, \ldots, N$ *the stopping times*

$$\sigma^*_{nN} = \min\{k \geq n : (1+r)^{-k} X_k = V^*_{kN} \text{ or } k = N\} \quad \textit{and} \tag{2.9}$$
$$\tau^*_{nN} = \min\{k \geq n : (1+r)^{-k} Y_k = V^*_{kN}\}$$

belong to $\mathcal{T}_{nN}$ *(since* $V^*_{NN} = (1+r)^{-N} Y_N$*) and they satisfy*

Theorem 2.1

$$E^*\left((1+r)^{-\sigma^*_{nN}\wedge\tau}R(\sigma^*_{nN},\tau)\middle|\mathcal{F}_n\right) \le V^*_{nN} \le E^*\left((1+r)^{-\sigma\wedge\tau^*_{nN}}R(\sigma,\tau^*_{nN})\middle|\mathcal{F}_n\right) \tag{2.10}$$

for any $\sigma, \tau \in \mathcal{T}_{n,N}$. *Finally, there exists a self-financing portfolio strategy* π^* *such that* (σ^*_{0N}, π^*) *is a hedge against this GCC with the initial capital* $Z_0^{\pi^*} = V^*_{0N}$ *and such strategy is unique up to the time* $\sigma^*_{0N} \wedge \tau^*_{0N}$.

Theorem 2.1

Proof. Let $\pi = (\pi_1, \ldots, \pi_N)$, $\pi_n = (\beta_n, \gamma_n)$ be a self-financing portfolio strategy with $Z_0^\pi = z > 0$ then $M_n^\pi = (1+r)^{-n} Z_n^\pi$ (see [SKKM1]),

$$M_n^\pi = z + \sum_{k=1}^{n} (1+r)^{-k} \gamma_k S_{k-1} (\rho_k - r), \qquad (2.11)$$

which is a martingale with respect to the filtration $\{\mathscr{F}_n\}_{0 \le n \le N}$ and the probability P^*.

(back)

Part1 of proof

- Suppose that (σ, π) is a hedge, by the Optional Sampling Theorem (see [Ne], Theorem II-2-13)

for any $\tau \in \mathcal{F}_{0N}$,
we have $Z_0^\pi = \mathrm{E}^*((1+r)^{-\sigma\wedge\tau} Z_{-\sigma\wedge\tau}^\pi)) \geq \mathrm{E}^*((1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)))$

Since, by the definition, V^* is the infimum of such initial capitals Z_0^π
then V^* is not less than the right hand side of (2.8).

Part2 of proof

- In the other direction

for any $\sigma \in \mathcal{F}_{0N}$,

set $V_n^\sigma = \max_{\tau \in \mathscr{T}_{nN}} E^*(U_\tau^\sigma | \mathscr{F}_n)$ where $U_k^\sigma = (1+r)^{-\sigma \wedge k} R(\sigma, k)$, $k = 0, 1, \ldots, N$.

Part2 of proof

- Observe that U_k^σ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable

It is easy to check directly and follows from general theorems (see [Ne], Proposition VI-1-2)

that $\{V_n^\sigma\}_{0\le n\le N}$ is a minimal supermartingale with respect to the filtration $\{\mathscr{F}_n\}_{0\le n\le N}$ such that $V_n^\sigma \ge U_n^\sigma$, $n = 0, 1, \ldots, N$.

Part2 of proof

- Proceeding in the standard way via the Doob supermartingale decomposition and the martingale representation

(obtain similarly to Sect. 2 and Sect. 5 in [SKKM1])

there exists a self-financing portfolio strategy $\pi^{\sigma} = (\pi_1^{\sigma}, \dots, \pi_1^{\sigma})$,

$\pi_n^{\sigma} = (\beta_n^{\sigma}, \gamma_n^{\sigma})$ with the portfolio value process $Z_n^{\pi^{\sigma}} = \beta_n^{\sigma} B_n + \gamma_n^{\sigma} S_n$

such that $(\pi, \sigma)$is a hedge.

Part3 of proof

- Next, define σ^*_{nN}, τ^*_{nN} by (2.9).

Then it is easy to see by the backward induction in n that (2.8) and (2.10) hold true.

Part4 of proof

Now take $\sigma^* = \sigma^*_{0N} \in \mathcal{T}_{0N}$ and construct the corresponding self-financing portfolio strategy $\pi^* = \pi^{\sigma^*}$, as above, which yields the hedge (σ^*, π^*) with the initial capital $V_0^{\sigma^*} = \max_{\tau \in \mathcal{T}_{0N}} E^*((1+r)^{-\sigma^* \wedge \tau} R(\sigma, \tau)) = V^*_{0N}$ where the last equality follows from (2.10). This together with the first part of the proof gives $V^* = V^*_{0N}$.

Part5 of proof

- It remains to obtain the uniqueness.

Set $\tau^* = \tau^*_{0N}$. Since $(\sigma^*, \pi^{\sigma^*})$ is a hedge

then $M_0^{\pi^{\sigma^*}} = V_0^{\sigma^*} = E^*((1+r)^{-\sigma^* \wedge \tau^*} R(\sigma^*, \tau^*)) \leq E^*((1+r)^{-\sigma^* \wedge \tau^*} Z^{\pi^{\sigma^*}}_{\sigma^* \wedge \tau^*}) =$
$E^* M^{\pi^{\sigma^*}}_{\sigma^* \wedge \tau^*} = M_0^{\pi^{\sigma^*}}$ since $M_n^{\pi^{\sigma^*}}$ is a martingale.

It follows that $Z^{\pi^{\sigma^*}}_{\sigma^* \wedge \tau^*} = R(\sigma^*, \tau^*)$.

Part5 of proof

- Let now $\pi = (\pi_1, \ldots, \pi_n), \pi_n = (\beta_n, \gamma_n)$ be another self-financing portfolio strategy with $Z_0^{\pi} = V^* = V_0^{\sigma^*}$.

According to the first part of the proof

$M_n^{\pi} = (1+r)^{-n} Z_n^{\pi}$ and $Z_{\sigma^* \wedge \tau^*}^{\pi} = \mathrm{R}(\sigma^*, \tau^*) = Z_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}}$

and so $\mathrm{M}_{\sigma^* \wedge \tau^*}^{\pi} = \mathrm{M}_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}}$

Part5 of proof

- Since both and M_n^{π}, $\mathrm{M}_n^{\pi^*}$ are martingales

it follows that $\mathrm{M}_n^{\pi} = \mathrm{M}_n^{\pi^*}$ and $Z_n^{\pi} = Z_n^{\pi^*}$ for all $n \leq \sigma^* \wedge \tau^*$

- Since the representation (2.11) is unique

$S_n > 0$ and $\rho_n \neq r$ for all n

then $\gamma_n = \gamma_n^{\pi^*}$ and $\beta_n = \beta_n^{\pi^*}$ for all $n \leq \sigma^* \wedge \tau^*$

Remark 2.2

- $R(\sigma,\tau)$ is replaced by $\hat{R}(\sigma,\tau) = X_\sigma \mathbb{I}_{\sigma<\tau} + Y_\tau \mathbb{I}_{\tau<\sigma} + W_\sigma \mathbb{I}_{\sigma=\tau}$ where W_n is $\mathcal{F}_n$-measurable, $Y_n \leq W_n \leq X_n$, n = 0, 1, . . . ,N and $W_N \leq Y_N$.

Remark 2.3

- Theorem 2.1 can be extended to the infinite horizon case $\mathrm{N} = \infty$

Remark 2.4

- Theorem 2.1 can be generalized to the case when consumption or infusion of capital is also possible.

$$Z^{\pi}_{n-1} = \beta_n B_{n-1} + \gamma_n S_{n-1} + g_n$$

$$V^* = \min_{\sigma \in \mathscr{T}_{0N}} \max_{\tau \in \mathscr{T}_{0N}} E^*((1+r)^{-\sigma\wedge\tau} R(\sigma,\tau) + \sum_{k=1}^{\sigma\wedge\tau} (1+r)^{-(k-1)} g_k$$

Remark 2.5

- It is easy also to generalize the above set up allowing dependence of r, a and b on time

i.e. assuming that $\rho_k(\omega) = \frac{1}{2}(a_k + b_k + \omega_k(b_k - a_k))$ and

$B_n = B_0 \prod_{k=1}^{n}(1 + r_k)$ where $r_k, a_k, b_k;\ k = 1, \ldots, N$ are nonrandom sequences

satisfying $-1 < a_k < r_k < b_k$.