A Multi-Phase, Flexible, and Accurate Lattice for Pricing Complex Derivatives with Multiple Market Variables*

Abstract

With the rapid growth and the deregulation of financial markets, many complex derivatives have been structured to meet specific financial goals. Unfortunately, most complex derivatives have no analytical formulas for their prices, particularly when there is more than one market variable. As a result, these derivatives must be priced by numerical methods such as lattice. However, the nonlinearity error of lattices due to the nonlinearity of the derivative's value function could lead to oscillating prices. To construct an accurate, multivariate lattice, this paper proposes a multi-phase method that alleviates the oscillating problem by making the lattice match the "critical locations," locations where nonlinearity of the derivative's value function occurs. Moreover, our lattice has the ability to model the jumps in the market variables like regular withdraws from an investment account, which is hard to deal with analytically. Numerical results for vulnerable options, insurance contracts guaranteed minimum withdrawal benefit (GMWB), and defaultable bonds show that our methodology can be applied to the pricing of a wide range of complex financial contracts.

Keywords: nonlinearity error, derivatives pricing, lattice, multi-asset

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1 Introduction

The financial markets have become vast, tightly integrated, and complex, particularly in the derivatives sector (see Bank for International Settlements). As a result, derivatives pricing has become much more complicated. One reason is that more sophisticated derivatives are constantly being structured to fit the needs of markets. Addressing their sophisticated features significantly increases the difficulty of pricing them (see Figlewski and Gao (1999)). Another reason is that the importance of some factors, like sovereign risk or credit risk, which are overlooked in earlier, primitive derivatives pricing models, is being recognized as key due to recent financial crises. If the default risk of the option issuer (or the counterparty risk) is ignored, analytical pricing formulas for vanilla options and barrier options have been derived by Black and Scholes (1973) and Reiner and Rubinstein (1991), respectively. However, when the structural model (see Merton (1974); Black and Cox (1976)) is introduced to model the counterparty risk, analytical formulas for the vulnerable vanilla options (defined in, e.g., Ammann (2001)) are only available under restricted assumptions (see Klein (1996); Klein and Inglis (2001)). In addition, no analytical formulas seem to exist for vulnerable barrier options under the Black and Cox (1976) first-passage model.

The lattice method and the closely related finite-difference method are popular and flexible numerical methods for pricing derivatives, particularly when analytical solutions do not exist or when analytical solutions involve multiple integrations. These methods can be easily modified to price derivatives with American and other exotic features. They divide the time dimension and the market factors/variables into discrete points. The results converge to the desired value under the continuous-time model as the number of discrete steps goes to infinity (see Duffie (1996)).

Figlewski and Gao (1999) point out that one of the most difficult issues facing lattice and finite-difference methods is the nonlinearity error. This error reveals itself as oscillating prices, resulting from the nonlinearity of the derivative's value function — like the kink of the option payoff function. Figlewski and Gao (1999) address this problem with the adaptive mesh model (AMM). The problem with the AMM is that its structure is complicated and hard to implement. Furthermore, it is difficult to tailor the AMM to many derivatives. Alternatively, the nonlinearity error can be reduced by restructuring the lattice to make a lattice's nodes, price levels, or time steps match the so-called "critical locations" — locations where nonlinearity of the derivative's value function occurs. Indeed, this idea is adopted by Ritchken (1995); Broadie and Detemple (1996); Cheuk and Vorst (1996, 1997); Tian (1999); Widdicks et al. (2002); Aitsahlia et al. (2004). However, these methods are not flexible enough for complex derivatives; besides, only one market variable is considered.

Multivariate derivatives elevate the pricing difficulty to a new level compared with that of univariate derivatives. First, the correlations between the market variables must be carefully handled because, otherwise, invalid branching probabilities may result (see Zvan et al. (2003)). Furthermore, Lyuu and Wang (2011) prove that the size of any valid constant-degree lattice for bivariate derivatives must explode when the second market variable is the interest rate that allows its value to grow lognormally without bound in magnitude, such as the popular model of Black et al. (1990).

One approach is to construct a multivariate lattice that matches the means, variances, and covariances of the market variables. For example, Rubinstein (1994) builds a three-dimensional lattice for two correlated assets by a non-rectangular arrangement of the lattice nodes. However, his lattice is not flexible enough to suppress the nonlinearity error. Another approach is to build a lattice by assuming that the processes of market variables are independent first, and then to adjust the branching probabilities to reflect the correlations (see Hull and White (1994)). However, the branching probabilities can become negative after the adjustments. Hull and White (1990b) suggest that the correlated processes can be first transformed into uncorrelated ones, and then an explicit finite-difference method, which is identical to the lattice, is built for these uncorrelated processes. However, the nonlinearity error problem remains unsolved. Andricopoulos et al. (2003) propose the quadrature method, which has a multinomial tree structure. It is extended by Andricopoulos et al. (2007) to handle multiple assets. This method can suppress the nonlinearity error by letting nodes or discrete time steps match the critical locations; in addition, it is efficient for pricing discretely monitored options, such as discrete barrier options, since it deploys only one time step between two monitoring dates. However, Dai and Lyuu (2010) show the quadrature method is not as efficient as the lattice in handling continuous sampling features, like the American exercise feature and the continuous barrier options. Besides, stochastic interest rate models, such as the Hull-White interest rate model (see Hull and White, 1990a), are not covered by their methods.

The major contribution of this paper is a general multi-phase method to build multivariate lattices for pricing complex derivatives without significant nonlinearity errors. By using the trinomial structure of Dai and Lyuu (2010) into our lattice, the length of a time step, the distance between two adjacent price levels, and even the position of the lattice can be changed to make the lattice match the critical locations. In addition, our lattice also generalizes the stair lattice proposed in Dai (2009); the stair lattice can handle the discrete price jumps in the underlying assets. This generalization significantly extends the capability of our lattice to handle real-world features that are hard to address, like asset sales to finance the discrete dividend payments and the insurance contract that allows the insurer to withdraw money from the account.

Hull and White (1990b) handle the correlations between market variables by the orthogonalization method, which transforms the original, correlated processes into uncorrelated ones. To simultaneously handle the correlations and make the lattice match the critical locations, we precede their orthogonalization method with ordering the market variables so that the i-th coordinates of the critical locations depend only on the first i of the market variables. The proposed multi-phase method builds the multivariate lattice for the transformed, uncorrelated processes with successive coordinates. The first phase constructs the lattice for the first uncorrelated process; this lattice will match the first coordinates of the critical locations. The second phase adds a lattice for the second uncorrelated process on top of the lattice constructed in the first phase to form a bivariate lattice while matching the second coordinates of the critical locations. A multivariate lattice is finally constructed by repeating the above method: The i-th phase adds a lattice for the i-th uncorrelated process on top of the $(i-1)$-variate lattice constructed in $(i-1)$-th phase to form an i-variate lattice. The i-th phase will adjust the lattice to match the i-th coordinates of the critical locations.

Take the pricing of vulnerable knock-out barrier options on stock under the first-passage model (see Black and Cox (1976)) as an example. There are two cases that the value of the option becomes zero with a zero recovery rate: (1) The stock price process reaches the barrier, and (2) the firm's asset value is less than the default boundary, which is the sum of the firm's outstanding bonds and the non-vulnerable option value. A bivariate lattice is constructed to describe the evolution of the two correlated market variables, i.e., the firm's asset value V and the stock price S. First, the two market variables are ordered: S is the first market variable and V the second. The reason for the choice is that the barrier for the stock price depends only on the value of S, whereas the default boundary for the firm's asset value depends on both S and V. We then transform the two correlated processes for S and V into two uncorrelated processes X and Y. The first phase constructs the lattice for the process X; this lattice will match the first coordinates of the critical locations, which only depend on the values of S. The second phase constructs a lattice for the process Y and combines it and the lattice for X to form a bivariate lattice; in the meantime, the second coordinates of the critical locations, depending on S and V, are also matched.

Our paper is organized as follows. The mathematical models and some preliminaries are introduced in Section 2. Section 3 describes the proposed multi-phase method to build multivariate lattices. Numerical results for vulnerable options, insurance contracts guaranteed minimum withdrawal benefit (GMWB), and defaultable bonds given in Section 4 show that our methodology can be applied to the pricing of a wide range of complex financial contracts. Section 5 concludes the paper.

2 Modeling and Preliminaries

2.1 The Lognormal Diffusion Process and the CRR Lattice

Define $S(t)$ as the value of a market variable at time t. $S(t)$ is said to follow a lognormal diffusion process if it can be represented as the following differential form under the risk-neutral probability:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dz(t), \tag{1}$$

where r is the risk-free rate, σ is the volatility of the market variable, and the random variable $dz(t)$ is the standard Brownian motion. The lognormal diffusion process is widely used to model the stock process (like Black and Scholes (1973)), the firm value (like Merton (1974)), and the portfolio value (like Dai et al. (2008); Wu (2009)). Equation (1) has the following solution (see Shreve (2004)):

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma z(t)}. \tag{2}$$

A lattice is a numerical pricing method that approximates a continuous-time stochastic process, e.g., $S(t)$ in Eq. (2). Let a derivative on $S(t)$ initiate at time 0 and mature at time T. A lattice partitions this time span into n equal-distanced time steps and specifies the value of $S(t)$ at each time step. The lattice converges to the continuous-time model (2) as $n \to \infty$ (see Duffie (1996)). Let the length between two adjacent time steps be $\Delta t \equiv T/n$. Take the popular Cox-Ross-Rubinstein

(CRR) binomial lattice of Cox et al. (1979) depicted in Fig. 1 for example. Each stock price S can either move upward to become Su with probability P_u, or move downward to become Sd with probability $P_d \equiv 1 - P_u$. The upward and downward movements (u and d) and the branching probabilities (P_u and P_d) of the CRR lattice are set to asymptotically match the mean (μ) and the variance ($\hat{\sigma}^2$) of the lognormal return of $S(t)$, which can be derived from Eq. (2) as

$$\begin{aligned} \mu &\equiv \left(r - \sigma^2/2\right) \Delta t, \\ \hat{\sigma}^2 &\equiv \sigma^2 \Delta t. \end{aligned}$$

The CRR lattice adopts the following solution:

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}}, \\ d &= e^{-\sigma\sqrt{\Delta t}}, \\ P_u &= \frac{e^{r\Delta t} - d}{u - d}, \\ P_d &= \frac{e^{r\Delta t} - u}{d - u}. \end{aligned}$$

For convenience, define the log-distance between stock prices S and S' as $|\ln(S) - \ln(S')|$. Thus the log-distance between any two adjacent stock prices at any given time step in the CRR lattice (see Fig. 1) is $2\sigma\sqrt{\Delta t}$.

2.2 The Trinomial Structure

Dai and Lyuu (2010) describe a trinomial structure that can be appended to a lattice that matches critical locations. We now review the trinomial structure for a single market variable $X(t)$. Let μ be the drift of $X(t)$ and 1 be the volatility. Consider an arbitrary node A. We shall construct a trinomial branching structure from node A with value $X(\tau)$ at time τ to three nodes at time $\tau + \Delta t'$ as illustrated in Fig. 2. The first two moments of $X(\tau + \Delta t')$ must be matched by the trinomial structure, and all the branching probabilities, P_u, P_m, and P_d, must be between 0 and 1. The nodes at time $\tau + \Delta t'$ are equally spaced with spacing $2\sqrt{\Delta t}$, and they are positioned so that one of the nodes at time $\tau + \Delta t'$ coincides with the critical location. Under the premise $\Delta t < \Delta t' \leq 2\Delta t$, node A is connected to the node (node C in Fig. 2) whose position is closest to the conditional expected value of $X(\tau + \Delta t')$, and node C's two adjacent nodes (nodes B and D). Define α, β, and γ as the subtractions of the conditional expected value of $X(\tau + \Delta t')$ from the values of nodes B, C, D, respectively. Dai and Lyuu (2010) prove that this yields valid branching probabilities:

$$
\begin{aligned}
P_u &= \frac{\beta\gamma + \Delta t'}{(\alpha - \beta)(\alpha - \gamma)}, \\
P_m &= \frac{\alpha\gamma + \Delta t'}{(\beta - \alpha)(\beta - \gamma)}, \\
P_d &= \frac{\alpha\beta + \Delta t'}{(\gamma - \beta)(\gamma - \alpha)}.
\end{aligned}
$$

2.3 The Hull-White Interest Rate Model and Its Lattice

The Hull-White interest rate model (see Hull and White (1990a)) is a no-arbitrage short rate model that is able to match the market's term structure of interest rates. The short rate at time t, $r(t)$, follows the stochastic process

$$dr(t) = (\theta(t) - ar(t))dt + \sigma_{\mathrm{r}} dz_{\mathrm{r}}, \tag{3}$$

where $\theta(t)$ is a function of time that makes the model fit the real-world interest rate market, a denotes the mean reversion rate for the short rate $r(t)$ to revert to $\theta(t)/a$, σ_{r} denotes the instantaneous volatility of the short rate, and dz_{r} is the standard Brownian motion. Hull and White (1994) provide a two-stage lattice-building procedure for the Hull-White interest rate model.

2.4 The Oscillation Problem and Nonlinearity Error

Although the prices generated by the lattice converge to the theoretical value of a contingent claim as $n \to \infty$ (see Duffie (1996)), they may oscillate significantly. According to Figlewski and Gao (1999), this phenomenon is due to the error introduced by the nonlinearity of the value function of the contingent claim. This nonlinearity error can be much reduced by making a node or a price level of the lattice match the critical locations where the value function of the contingent claim is highly nonlinear (see Dai and Lyuu (2010)). Critical locations are straightforward to identify. For example, the vanilla option has a critical point at maturity with the stock price equal to the strike price. For continuous barrier options, the critical price level occurs along the barrier price.

For pricing vulnerable options and defaultable bonds under the structural model, critical locations run along the exogenous default boundary. The hard problem is that, in general, the positions of critical locations on a multivariate lattice can depend on multiple market variables; for example, the default boundary on the firm's asset value for a defaultable bond under the stochastic interest rate model depends on both the firm's asset value and interest rate. Therefore, it is hard for a lattice to match these critical locations by the traditional construction methodology.

3 Lattice Construction

For the sake of brevity, we demonstrate our multi-phase construction method with a bivariate lattice for two correlated market variables. The construction for lattices for more than two market variables shares the same methodology. To handle the correlations between the market variables, we rely on the orthogonalization to transform the original, correlated processes into uncorrelated ones; the orthogonalization process is described in Subsection 3.1. Subsection 3.2 describes the core idea of our multi-phase branch construction by an example. Subsection 3.3 shows how to construct a bivariate lattice for two correlated lognormal processes, the underlying being the stock price and the firm's asset value, to price vulnerable options. The trinomial structure discussed in Subsection 2.2 is applied to make the bivariate lattice match barriers and default boundaries. A 2-time-step lattice is given to convey main ideas. Our lattice construction methodology can also be used to build multivariate lattices for other market variables. Subsection 3.4 proves that by building a bivariate lattice with asset price and interest rate as market variables. When the underlying asset is interpreted as the firm's asset value, this lattice can be used to evaluate interest-sensitive securities on the firm, like defaultable corporate bonds. When the underlying asset is interpreted as the insurer's account, this lattice can be used to evaluate variable annuity, like GMWB. GMWB allows the insurer to withdraw a fixed amount of money from the insurer's account at withdrawal dates, and this covenant results in a downward jump of the value of the insurer's account. Our method can deal with such jumps without difficulty.

3.1 Transforming Correlated Processes into Uncorrelated Ones via Orthogonalization

The orthogonalization is a method that converts a nonorthogonal set of linearly independent functions to an orthogonal basis. With this method, we can transform a set of correlated processes into a set of uncorrelated ones. For the sake of brevity, we demonstrate this transformation for a set of two correlated processes. The transformation for more than two correlated processes shares the same methodology.

To simultaneously handle the correlations and make our lattice match the critical locations, we revise Hull and White's (1990b) orthogonalization method, which transforms the original, correlated processes into uncorrelated ones. Before the transformation, the market variables are so ordered that the i-th coordinates of the critical locations depend only on the first i of the market variables. For example, the two correlated market variables are ordered so that S_1 is followed by S_2 when the first coordinates of the critical locations are functions of S_1 and the second coordinates are functions of S_1 and S_2.

Let S_1 and S_2 be represented as follows:

$$\begin{aligned} dS_1 &= \mu_1 dt + \sigma_1 dz_1, \\ dS_2 &= \mu_2 dt + \sigma_2 dz_2. \end{aligned}$$

Above, for $x \in \{1, 2\}$, μ_x denotes the drift of S_x, σ_x denotes the volatility of S_x, and dz_x denotes the standard Brownian motion. The correlation between dz_1 and dz_2 is ρ.

It is well-known that dz_2 can be decomposed into a linear combination of dz_1 and another independent Brownian motion dz thus:

$$dz_2 = \rho\, dz_1 + \sqrt{1-\rho^2}\, dz. \tag{4}$$

The differential forms of S_1 and S_2 can be represented in the following matrix form,

$$\begin{bmatrix} dS_1 \\ dS_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz \end{bmatrix}. \tag{5}$$

For this two-asset case, define A as

$$A = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{bmatrix}. \tag{6}$$

Thus the inverse matrix A^{-1} equals

$$A^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ \frac{-\rho}{\sigma_1 \sqrt{1-\rho^2}} & \frac{1}{\sigma_2 \sqrt{1-\rho^2}} \end{bmatrix}. \tag{7}$$

We proceed to transform S_1 and S_2 into two uncorrelated processes X_1 and X_2. The differential forms of X_1 and X_2 can be obtained by multiplying A^{-1} to both sides of Eq. (5):

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} dS_1/\sigma_1 \\ \frac{-\rho\, dS_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{dS_2}{\sigma_2\sqrt{1-\rho^2}} \end{bmatrix} = \begin{bmatrix} \mu_1/\sigma_1 \\ \frac{-\rho\, \mu_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{\mu_2}{\sigma_2\sqrt{1-\rho^2}} \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dz_1 \\ dz \end{bmatrix}. \tag{8}$$

Integrate both sides of Eq. (8) to yield

$$X_1(t) = \frac{S_1(t) - S_1(0)}{\sigma_1}, \tag{9}$$

$$\begin{aligned} X_2(t) &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{S_2(t) - S_2(0)}{\sigma_2} - \rho\, \frac{S_1(t) - S_1(0)}{\sigma_1} \right) \\ &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{S_2(t) - S_2(0)}{\sigma_2} - \rho X_1(t) \right), \end{aligned} \tag{10}$$

where $X_1(0) = X_2(0) = 0$ for convenience. The market variables $S_1(t)$ and $S_2(t)$ can be expressed in terms of $X_1(t)$ and $X_2(t)$:

$$S_1(t) = S_1(0) + \sigma_1 X_1(t), \tag{11}$$

$$S_2(t) = S_2(0) + \sigma_2 \left(\sqrt{1-\rho^2} X_2(t) + \rho X_1(t) \right). \tag{12}$$

The above method can be easily generalized to handle more than two correlated stochastic processes. If n correlated processes need to be orthogonalized, the lower-triangular matrix A (e.g., Eq. (6)) can be obtained by decomposing the covariance matrix of n correlated Brownian motions by the Cholesky decomposition (see Golub and Van Loan (1996)).

Recall that our lattice is built to approximate the uncorrelated processes $X_1(t)$ and $X_2(t)$; the values of the original market variables ($S_1(t)$ and $S_2(t)$) on each node of the lattice can be recovered by substituting $X_1(t)$ and $X_2(t)$ into Eqs. (11)–(12). Note that the fact that $S_1(t)$ depends only on $X_1(t)$, and vice versa, can help locate the positions of critical locations and adjust the lattice to match the locations.

3.2 Core Ideas of Our Multi-Phase Branch Construction

The core ideas of our multi-phase branch construction are illustrated in Fig. 3. Consider a bivariate lattice that approximates the evolution of two uncorrelated processes $X_1(t)$ and $X_2(t)$. The construction contains two phases. Let us focus on node A with value $(X_1(\tau), X_2(\tau))$ in Fig. 3. The first phase constructs the trinomial structure from node A', the projection of node A on the X_1-t plane, to approximate process $X_1(t)$. After selecting nodes B', C', and D' as the destinations of the trinomial structure, the branching probabilities P_u, P_m, and P_d are calculated. The second phase first approximates $X_2(t)$; the trinomial structure from node A to nodes B_1, B_2, and B_3 is constructed, and the corresponding probabilities Q_u, Q_m, Q_d are calculated. After combining the branches from the two planes, node A now has nine branches, and the joint branching probabilities can be obtained by multiplying the branching probabilities in the X_1 dimension and those in the X_2 dimension. For example, the joint branching probability from node A to node B_1 is $P_u Q_u$. Note that the nodes at time $\tau + \Delta t'$ on both X_1-t and X_2-t planes are positioned so that one of the nodes at time $\tau + \Delta t'$ coincides with the critical location (see the next section for details on how to match the critical locations).

3.3 A Bivariate Lattice: Two Correlated Market Variables

This subsection focuses on the bivariate lattice for two market variables: the stock price, $S(t)$, and the firm's asset value, $V(t)$. The bivariate lattice is built to price vulnerable barrier options with the strike price K and the barrier $B(t) = Be^{-\gamma(T-t)}$. The default boundary for the firm's asset value at time t, $D^*(t)$, is assumed to be the sum of the non-vulnerable option value at time t, $c(S(t), t)$, and the discounted value of the firm's outstanding debt; that is, $D^*(S(t), t) = De^{-r(T-t)} + c(S(t), t)$, where D denotes the firm's outstanding debt (see Klein and Inglis, 2001). In this setup, the option holder receives

$$c(S(t),t)/D^*(S(t),t) \tag{13}$$

of the firm's asset value when the firm defaults.

The two market variables, $S(t)$ and $V(t)$, are both assumed to follow the processes in Eq. (1):

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= rdt + \sigma_S dz_S, \\
\frac{dV(t)}{V(t)} &= rdt + \sigma_V dz_V.
\end{aligned}$$

By Ito's lemma,

$$\begin{aligned}
d\ln S(t) &= \left(r - \frac{\sigma_S^2}{2}\right) dt + \sigma_S dz_S, \\
d\ln V(t) &= \left(r - \frac{\sigma_V^2}{2}\right) dt + \sigma_V dz_V, \qquad (14)
\end{aligned}$$

where r is the risk-free rate, σ_S and σ_V are the volatilities of $S(t)$ and $V(t)$, respectively, and dz_S and dz_V are two correlated standard Brownian motions with correlation ρ.

We first order the original two processes in Eq. (14); $\ln S(t)$ is the first process and $\ln V(t)$ the second. The reason for the choice is that the barrier for the stock price is $f(\ln(S(t)))$, whereas the default boundary for the firm's asset value is $g(\ln S(t), \ln V(t))$ (see Subsection 3.1). We then apply the orthogonalization process in Subsection 3.1 to the processes, to obtain two uncorrelated processes, $X(t)$ and $Y(t)$. According to matrix equation (8),

$$\begin{aligned}
dX(t) &= \frac{1}{\sigma_S}\left(r - \frac{\sigma_S^2}{2}\right) dt + dz_S, \qquad (15)\\
dY(t) &= \frac{1}{\sqrt{1-\rho^2}}\left(-\frac{\rho}{\sigma_S}\left(r - \frac{\sigma_S^2}{2}\right) + \frac{1}{\sigma_V}\left(r - \frac{\sigma_V^2}{2}\right)\right) dt + dz, \qquad (16)
\end{aligned}$$

where dz_S and dz are uncorrelated.

The bivariate lattice is built from two lattices, one for $X(t)$ and another for $Y(t)$, in the following way. Each node on a bivariate lattice at time step j pairs a value from the lattice for $X(t)$ with a value from the lattice for $Y(t)$, both at time step j. Since $X(t)$ and $Y(t)$ are uncorrelated, the branching probabilities can be obtained by multiplying the branching probabilities of the $X(t)$-

lattice and those of the $Y(t)$-lattice.

The construction for the bivariate lattice has two phases. In the first phase, the $X(t)$-lattice is built. By Eq. (9), the barrier $B(t)$ for the stock price $S(t)$ can be transformed to the barrier $B_X(t)$ on the $X(t)$-lattice:

$$B_X(t) = \frac{1}{\sigma_S} \left(\ln B(t) - \ln S(0) \right). \tag{17}$$

Figure 4 depicts a 3-time-step $X(t)$-lattice with the time-varying barrier $B_X(t)$. The lattice starts by placing gray nodes on the barrier to reduce the nonlinearity error. All the other nodes are then laid from the gray nodes upward and downward with the log-distance between any two vertically adjacent nodes being $2\sqrt{\Delta t}$. This setting helps us construct trinomial branches from any node at time step t to the three successor nodes at time step $t+1$ via the procedure in Subsection 2.2. Start from node X for $X(0)$. Its successor nodes will be selected from the nodes at time step 1, the successor nodes of these 3 nodes will be selected from the nodes at time step 2, and so on. Note that the option knocks out once $X(t)$ reaches the gray nodes on the barrier. The gray nodes, therefore, have no need for successor nodes.

We now proceed to the second phase, which builds the $Y(t)$-lattice at first. Recall that the default boundary for the firm's asset value $V(t)$ is assumed to be $D^*(t) = De^{-r(T-t)} + c(S(t), t)$. The term $c(S(t), t)$ can be evaluated by the Black-Scholes formula as follows:

$$c(S(t), t) = S(t)\mathbf{N}(d_1) - Ke^{-r(T-t)}\mathbf{N}(d_2), \tag{18}$$

where $\mathbf{N}(\cdot)$ denotes the CDF of the standard, 1-dimensional normal distribution, and

$$\begin{aligned} d_1 &= \frac{\ln(S(t)/K) + (r + \sigma_S^2/2)(T-t)}{\sigma_S\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma_S\sqrt{T-t}. \end{aligned}$$

By Eq. (10), the default boundary $D^*(t)$ for the firm's asset value can be transformed to the boundary $D_Y^*(t)$ on the $Y(t)$-lattice thus:

$$D_Y^*(t) = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln D^*(t) - \ln V(0) - \rho X(t)}{\sigma_V} \right). \tag{19}$$

Once we have $D_Y^*(t)$, the lattice again starts by placing nodes on this default boundary (the black nodes in Fig. 5). Note that each node with the value x on the $X(t)$-lattice corresponds a unique stock price $S(0)e^{\sigma_S x}$ by Eq. (11). Since $D_Y^*(t)$ depends on the stock price as in Eq. (19), each node on the $X(t)$-lattice will map to a value of $D_Y^*(t)$ given the value of this node. For example, in Fig. 5, node a on the $X(t)$-lattice corresponds to node A on the default boundary; similarly, node b corresponds to node B on the default boundary. The branch construction procedure for the $Y(t)$-lattice is identical to that for the $X(t)$-lattice.

In the end of the second phase, the $Y(t)$-lattice is added on top of the $X(t)$-lattice to form the bivariate lattice. Recall that each node in the bivariate lattice corresponds to a bivariate state with a value from the $X(t)$-lattice and a value from the $Y(t)$-lattice. As the trinomial $X(t)$-lattice and the trinomial $Y(t)$-lattice are combined to build our bivariate lattice, there are $3 \times 3 = 9$ branches per node as illustrated in Fig. 6. Node X at time step t has 9 branches, to nodes A, B, C, D, E, F, G, H, and I at time step $t+1$. It is straightforward to show that the size of the bivariate lattice is $O(n^3)$.

An Example Lattice

This subsection provides a numerical example to illustrate how to construct a bivariate lattice for pricing vulnerable barrier options. Assume the strike price is $K = 30$, the maturity is $T = 3$, the initial stock price is $S_0 = 40$, the volatility of the stock price is $\sigma_S = 20\%$, the firm's initial asset value is $V_0 = 100$, the volatility of the firm's asset value is $\sigma_V = 20\%$, the correlation between the stock price and the firm's asset value is $\rho = 0$, the risk-free interest rate is $r = 5\%$, and the recovery rate is $\alpha = 0.75$. The barrier is $B(t) = Be^{-\gamma(T-t)}$, where $B = 35$ and $\gamma = 0.01$, and the default boundary for the firm's asset value is assumed to be $D^*(t) = De^{-r(T-t)} + c(S(t), t)$, where $D = 90$.

The construction for the $X(t)$-lattice is illustrated in Fig. 7. By Eq. (17), the barrier $B(t)$ for the stock price $S(t)$ can be transformed to the barrier $B_X(t)$ on the $X(t)$-lattice. For example,

$$B_X(1.5) = \frac{\ln(35 \times e^{-0.01 \times 1.5}) - \ln(40)}{0.2} \approx -0.743.$$

Once the barrier $B_X(t)$ at each time step is obtained, we place gray nodes at Δt apart on the barrier to reduce the nonlinearity error. All the other nodes are laid from the gray nodes upward and downward with the log-distance between any two vertically adjacent nodes being $2\sqrt{\Delta t} = 2.45$. The branching probabilities for nodes A and B can be calculated by the procedure in Subsection 2.2.

We now start to build the $Y(t)$-lattice. We first recover the value of $S(t)$ at each node on the $X(t)$-lattice by Eq. (11). Then the term $c(S(t), t)$ in the default boundary for the firm's asset value, $D^*(t) = De^{-r(T-t)} + c(S(t), t)$, can be obtained by Eq. (18); we therefore have $D^*(t)$ for each node on the $X(t)$-lattice. For example, with $S(1.5) = 56.275$ (i.e., the stock price of node B), we have

$$D^*(1.5) = 90 \times e^{-0.05 \times 1.5} + 28.019 \approx 111.516. \tag{20}$$

Note that $c(S(t), t) = 0$ when the option knocks out (i.e., the gray nodes in Fig. 7). By Eq. (19), the default boundary $D^*(t)$ for the firm's asset value can be transformed to the boundary $D_Y^*(t)$ on the $Y(t)$-lattice. For example, by the result in Eq. (20),

$$D_Y^*(1.5) = \frac{1}{\sqrt{1-0}} \left(\frac{\ln(111.516) - \ln(100)}{0.2} - 0 \times 1.707 \right) \approx 0.545.$$

Once we have $D_Y^*(t)$, all the nodes on the $Y(t)$-lattice are laid on top of $D_Y^*(t)$ and then from the $D_Y^*(t)$ upward and downward as illustrated in Fig. 8. In Fig. 8, the firm defaults when $Y(t)$ reaches the dark-gray nodes, which, therefore, have no need for successor nodes. The branching probabilities can be obtained by the procedure in Subsection 2.2.

We now calculate the value of the vulnerable option on the 2-time-step bivariate lattice in Fig. 8 via backward induction. At maturity (time step 2), the option knocks out when the stock price hits the barrier (the light-gray node), so the option value is 0 (node E). If the option does not knock out (i.e., nodes A, B, C, and D), there are two cases: (1) the firm defaults if its asset value is less than the default boundary $D_Y^*(t)$; (2) otherwise, the firm survives (the white nodes). For example, at node c_1, the firm survives and the option value is $(93.237 - 30, 0)^+ \approx 63.24$. As another example, the firm defaults at node d_3, and the option value is

$$0.75 \times 117.125 \times \frac{(57.125 - 30)^+}{117.125} \approx 20.344$$

by Eq. (13). The option values at time step 1 are obtained by backward induction with the appropriate branching probabilities. For example, the value of the option at node b_1 equals

$$\begin{aligned} e^{0.05\times 1.5} \quad \times \quad & (0.157 \times (0.445 \times 63.24 + 0.554 \times 47.43 + 0.002 \times 29.06) + \\ & 0.746 \times (0.121 \times 27.13 + 0.75 \times 27.13 + 0.129 \times 20.34) + \\ & 0.096 \times 0) \approx 26.13. \end{aligned}$$

Note that the joint probabilities can be obtained by multiplying the branching probabilities on the $X(t)$-lattice and the $Y(t)$-lattice because $X(t)$ and $Y(t)$ are uncorrelated. Finally, the option value for node A at year 0 can be computed thus:

$$\begin{aligned} e^{0.05\times 1.5} \quad \times \quad & (0.401 \times (0.068 \times 26.13 + 0.733 \times 21.02 + 0.199 \times 12.88) + \\ & 0.594 \times 0 + 0.006 \times 0) \approx 7.337. \end{aligned}$$

3.4 A Bivariate Lattice with Stochastic Interest Rate as the Second Market Variable

This subsection focuses on bivariate lattices with the following two market variables: the interest rate, $r(t)$, and the firm's asset value, $V(t)$. The asset value is assumed to follow the process in Eq. (1). The bivariate lattice is built to price defaultable bonds with a positive net-worth covenant $\xi PV(t)$ as the default boundary (see Black and Cox (1976)) under the Hull-White interest rate model in Eq. (3), where $PV(t)$ denotes the sum of the present values of the unpaid coupons and the face value at maturity of the defaultable bond at time t, where ξ is a constant.

We first apply the orthogonalization process mentioned in Subsection 3.1 to the original two processes, $r(t)$ and $V(t)$ in Eq. (3) and Eq. (14), respectively, to obtain two uncorrelated process $X(t)$ and $Y(t)$. According to matrix equation (8),

$$\begin{aligned} dX(t) &= \frac{\theta(t) - ar(t)}{\sigma_{\mathrm{r}}} dt + dz_{\mathrm{r}}, \\ dY(t) &= \frac{1}{\sqrt{1-\rho^2}} \left(-\frac{\rho}{\sigma_V} \left(\theta(t) - ar(t)\right) + \frac{1}{\sigma_V} \left(r(t) - \frac{\sigma_V^2}{2} \right) \right) + dz, \end{aligned}$$

where dz_{r} and dz are uncorrelated.

With $X(t)$ and $Y(t)$ in place, the construction procedure for the bivariate lattice follows that in Subsection 3.3. For the $X(t)$-lattice, we have $X(t) = (r(t) - r(0))/\sigma_{\mathrm{r}}$ by Eq. (9). Since $X(t)$ is a bijection, we first construct the lattice for the interest rate by the procedure in Subsection 2.3 and then transform this lattice to form the $X(t)$-lattice. For the $Y(t)$-lattice, by Eq. (10), the default boundary $PV(t)$ can be transformed to the boundary on the $Y(t)$-lattice. Once we have the default boundary on the $Y(t)$-lattice, the construction methodology is the same as that in Subsection 3.3.

4 Numerical Evaluation

This section shows that our methodology can be applied to the pricing of a wide range of complex financial contracts, such as vulnerable options, insurance contracts GMWB, and defaultable bonds. We first analyze the numerical results for vulnerable vanilla and barrier options with a bivariate lattice for two correlated lognormal processes, the underlying being the stock price and the firm's asset value. Our methodology can also build bivariate lattices with asset price and interest rate as the market variables. When the underlying asset is interpreted as the firm's asset value, this lattice can be used to evaluate interest-sensitive securities on the firm, like defaultable corporate bonds. When the underlying asset is interpreted as the insurer's account, this lattice can be used to evaluate variable annuity, like GMWB. Subsection 4.2 analyzes the numerical results for the defaultable corporate bonds and GMWB. The following numerical results confirm that our lattice can generate accurate and smoothly-convergent prices that are unavailable to alternative analytical formulas and numerical methods.

4.1 Vulnerable Options

Many financial institutions trade derivatives through the over-the-counter (OTC) markets with other financial institutions or their corporate clients. Since there are no margin and daily settlement mechanisms in the OTC markets, the holders of these financial derivatives are vulnerable to counterparty credit risk (see Johnson and Stulz, 1987). These derivatives are sometimes called vulnerable derivatives (see Ammann (2001)). The solvency of the counterparty must be considered to evaluate them. The structural model pioneered by Merton (1974) is adopted here to model the financial status of the counterparty. In this subsection, the counterparty is assumed to be a firm, the underlying asset is assumed to be stock, and the derivative is assumed to be option for convenience. To evaluate a vulnerable option, our bivariate lattice models the dynamics of the firm's asset value and the stock price. It has the capability to fit the settings of different structural models to suppress the numerical errors.

Table 1 tabulates the values of the vulnerable vanilla options under different structural models. To make analytical formulas possible, Klein (1996) and Klein and Inglis (2001) make some simplifying assumptions. Both papers follow Merton (1974) by assuming that the firm can only default at the option's maturity date, and it defaults when it cannot meet the debt obligation then. The debt obligation in Klein (1996) is approximately set as the value of all other outstanding debts of the firm, D, by assuming that the obligation of the option is much smaller than D. The option holder receives $c(S(T),T)/D$ of the firm's asset value when the firm defaults. On the other hand, Klein and Inglis (2001) include the potential liability of the written option into the debt obligation so the total obligation at maturity becomes $D+c(S(T),T)$. In this case, the option holder receives $\frac{c(S(T),T)}{D+c(S(T),T)}$ of the firm's asset value when the firm defaults. The prices generated by their formulas appear in the second and forth columns of the table. The two columns marked by `Lattice` next to the aforementioned two `Formula` columns denote the prices generated by our lattice. The accuracy of our lattice is verified by observing that the prices generated by our lattice are close to those generated by the analytical formulas. Note that a higher debt obligation implies a higher likelihood for option holders to suffer from the loss of firm's default at the option's maturity date. Thus the option prices under the Klein and Inglis (2001) setting (with a higher debt obligation $D + c(S(T),T)$) should be lower than those under the Klein (1996) setting (with a lower debt obligation D), as expected.

The first-passage model (FPM hereafter) proposed by Black and Cox (1976) generalizes Merton's credit risk model by allowing the default to occur prior to the option's maturity. In the sixth column, we generalize the setting in Klein (1996) by assuming that the default boundary at time t as the discounted value of D; that is, $De^{-r(T-t)}$. In the seventh column, the setting in Klein and Inglis (2001) is generalized by assuming that the default boundary as the sum of the non-vulnerable option value at time t, $c(S(t),t)$, and the discounted value of other outstanding debts; that is, $De^{-r(T-t)} + c(S(t),t)$. In this case, analytical formulas and numerical methods are hard to come by since the default boundary contains a stochastic term $c(S(t),t)$, which is governed by the stock price process. Again, a higher default boundary implies a higher likelihood for the option holders to suffer from the loss of firm's default. Thus under the FPM, the option prices under the generalized Klein and Inglis (2001) setting (with a higher default boundary $De^{-r(T-t)} + c(S(t), t)$) should also be lower than those under the Klein (1996) setting (with a lower default boundary $De^{-r(T-t)}$), as expected.

Our lattice can be extended to price barrier options with counterparty risk. To suppress the nonlinearity error, our lattice matches both the default boundary in the firm's asset value and the barrier in the stock price. The robustness and the fast convergence of our lattice is confirmed by the solid line in Fig. 9 for pricing a vulnerable barrier call option under the FPM. To our knowledge, no analytical formulas are available to handle this problem in which each dimension has one boundary (the default boundary and the barrier); even both boundaries are all simple constants. A naive numerical method that does not align with the barrier and the default boundary will generate oscillating prices depicted as the dashed line in Fig. 9.

Our lattice is able to price vulnerable barrier options under various structural models and different barrier settings as illustrated in Table 2. Analytical formulas are only available for Merton's structural model with a constant debt obligation (see Pan (2010)) as in the `Formula` column. The robustness of our lattice is corroborated by the closeness of the prices generated by our lattice to those generated by analytical formulas. Furthermore, our lattice can be easily extended to deal with other more complex structural models that are analytically intractable. Like the results of the vulnerable vanilla options in Table 1, the values of the vulnerable barrier options priced under the setting in Klein and Inglis (2001) are lower than those priced under the setting in Klein (1996).

Option price behaviors may differ greatly under different structural models. In the volatility sensitivity plot in Fig. 10, for example, the discrepancy of option prices under two different structural models increases with the stock price volatility. Moreover, the option values under Merton's model with the Klein and Inglis (2001) setting assume a humped shape, while the option values under other models increase monotonically with volatility.

The relationships between option values and structural models are discussed as follows. Since the default boundary under Klein (1996) is lower than that under Klein and Inglis (2001), the firm incurs less bankruptcy cost under the former setting than the latter one. Therefore, the option values under Klein (1996) (the dotted and dashed lines in Fig. 10) are greater than those under Klein and Inglis (2001) (the solid and dot-dashed lines). Compared to Merton's model, the FPM allows the firm to default prior to maturity and has a higher default probability. Thus more bankruptcy cost is introduced under the FPM, which harms the value of the vulnerable option. However, the FPM provides protections to option holders by letting the firm default when its asset value goes below the default boundary. This protection raises the option value. If the effct of incurring more bankruptcy cost under the FPM dominates that of the protection provided by the FPM, the option values under Merton's model will be higher than those under the FPM; see the prices in Fig. 10 under the Klein (1996) setting for example. On the contrary, Klein and Inglis (2001) include the potential liability of the written option ($c(S(t), t)$) into the debt obligation which provides more protection to the option holders. So in this case, the effect of the protection dominates that of increased bankruptcy cost and the option values under Merton's model are lower than those under the FPM.

4.2 Evaluating Defaultable Bonds and GMWBs under Short Rate Models

To price defaultable bonds under short rate models, we build a bivariate lattice with asset price and interest rate as market variables. Our lattice describes the dynamics of the asset value and the interest rate, which is assumed to follow the Hull-White model (see Hull and White (1994)). Table 3 compares defaultable zero-coupon bond prices by our lattice and those by the analytical formula proposed by Briys and De Varenne (1997). The robustness of our lattice is affirmed by the low relative pricing errors. Note that the values of defaultable zero-coupon bonds are lower than default-free ones due to the credit risk. Their price discrepancy increases with the face value since the credit risk increases with the amount of outstanding bonds.

Both asset-sales assumptions and positive net-worth covenants affect bond prices. The coupon payments can be either financed by selling the firm's asset, which is called the total-asset-sales assumption (see Merton (1974); Brennan and Schwartz (1978)), or by issuing new equities, which is called the no-asset-sales assumption (see Leland (1994)). Lando (2004) argues that the existence of jumps under the total-asset-sales assumption makes analytical formulas impossible and lattices non-recombining. But our proposed methodology proves otherwise. The defaultable bond prices under these two assumptions are listed under the "Total-asset-sales" and "No-asset-sales" columns in Table 4. Under the total-asset-sales assumption, the firm's asset value jumps downward by the amount of the coupon at the payment date. These downward jumps weaken the firm's financial status and increase its default risk. That explains why the bond values under the total-asset-sales assumption are lower than those of an otherwise identical bond under the no-asset-sales assumption. The only exception when the bonds values are equal is the annual coupon payment setting. This is because the only coupon is paid at maturity T=1 (year), which, coinciding with the return of the face value, can be treated as part of the face value.

The FPM assumes the firm is allowed to default prior to maturity and the firm defaults when its asset value goes below an exogenous default boundary. We follow Leland (1994) by calling a bond with an exogenous default boundary a protected bond, and a bond without an exogenous default boundary an unprotected one. In numerical experiments, we adopt Briys and De Varenne (1997) in setting the boundary at $\xi = 90\%$ of the present value of bonds' unpaid coupons and the face values at maturity. Note that this stochastic default boundary depends on the term structure of interest rates, and the naive lattice will find it hard to address the price oscillation problem. In contrast, our lattice can address this problem by the technique in Subsection 3.3. As shown in Table 4, the above-mentioned default boundary does affect the prices of the bonds.

Although the early redemption features (such as callable or putable rights) are hard to evaluate analytically, our bivariate lattice can easily cope with it, as illustrated in Table 5. As expected, the prices of defaultable bonds are always lower than those of default-free bonds regardless of the presence of early redemption features. It is also not surprising that the callable feature decreases the bond value, whereas the putable feature increases it. Finally, the values of both callable and putable bonds increase with the call and put prices, as expected.

The aforementioned bivariate lattice for evaluating defaultable bonds can be extended to evaluate GMWBs by replacing the firm's asset value with the policy holder's account value. The insurer will receive the charge from the policy holder to make the expected present value of what the policy holder can receive from the insurer equal to the initial payment to the insurer. Thus, to evaluate the charge accurately is a vital issue. Table 6 lists the fare charges of the GMWBs under different maturities and correlations between the account value and the interest rate. Observe that the charges decrease with maturities but increase with correlations.

5 Conclusions

This paper proposes a flexible multi-phase method to build a multivariate lattice for pricing derivatives accurately. To simultaneously handle the correlations and make the lattice match the critical locations, the market variables are first properly ordered. Then our multivariate lattice describes the evolution of the uncorrelated processes transformed from the original, correlated processes of market variables by orthogonalization. Numerical results for vulnerable options, insurance contracts GMWB, and defaultable bonds show that our multi-phase method can be applied to the pricing of various complex financial contracts accurately.

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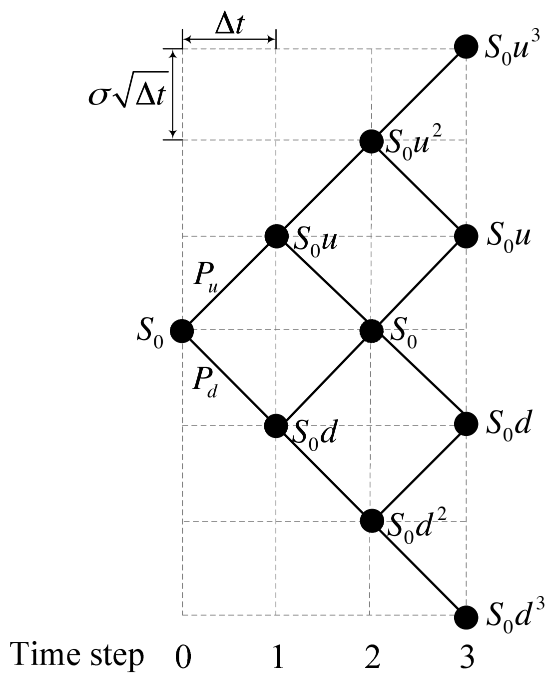

Figure 1: **The CRR Lattice.** The value of process $S(t)$ at time step 0 is denoted as S_0. The upward and downward multiplicative factors are u and d, respectively. The upward and downward branching probabilities are P_u and $P_d = 1-P_u$, respectively. The log-distance between two adjacent stock prices at any given time step is $2\sigma\sqrt{\Delta t}$, where Δt is the length of a period.

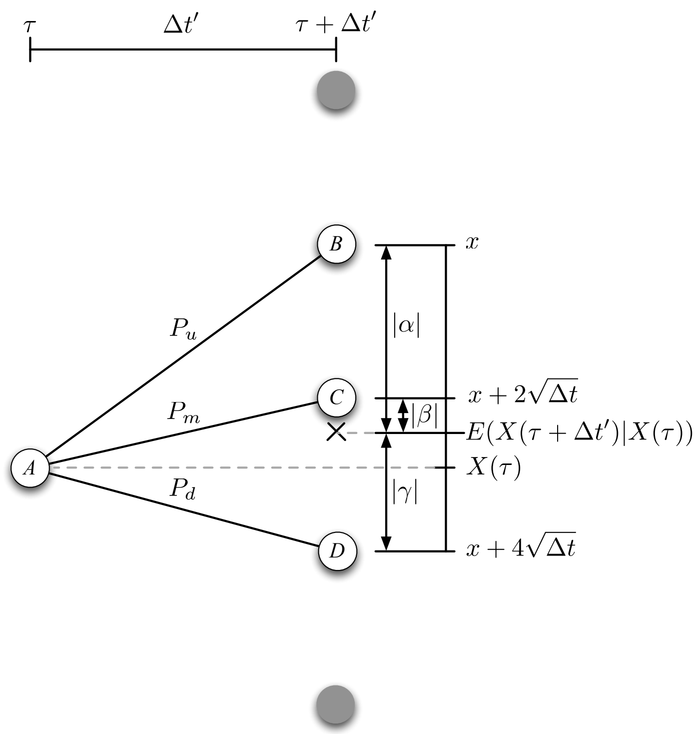

Figure 2: **The Trinomial Structure.** Node A with value $X(\tau)$ at time τ can move to node B(at position x) with probability P_u, node C (at position $x + 2\sqrt{\Delta t}$) with probability P_m, and node D (at position $x + 4\sqrt{\Delta t}$) with probability P_d. The conditional expected value of $X(\tau + \Delta t')$, or $E(X(\tau + \Delta t')|X(\tau))$, is marked by the cross. The symbols $|\alpha|$, $|\beta|$, and $|\gamma|$ denote the distances between $E(X(\tau + \Delta t')|X(\tau))$ and the nodes B, C, D, respectively.

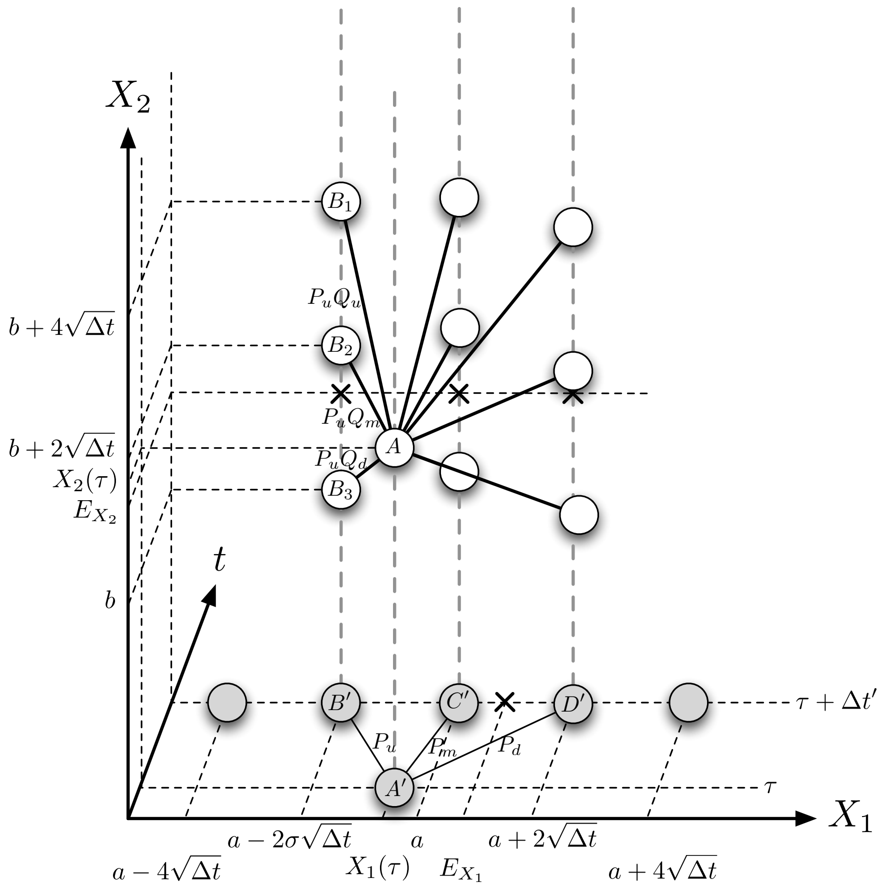

Figure 3: **The Branching Structures of Our Bivariate Lattice.** The branching structure from node A at time τ to the nine nodes at time $\tau + \Delta t'$ are marked in thick black lines. The projections of nodes A, B_1, ... are marked by dashed gray lines. The trinomial branches emitting from node A' to nodes B', C', and D' with probabilities P_u, P_m, and P_d approximate the evolution of the process X_1. E_{X_1} and E_{X_2} denote the conditional expected values of processes X_1 and X_2 at time $\tau + \Delta t'$, respectively.

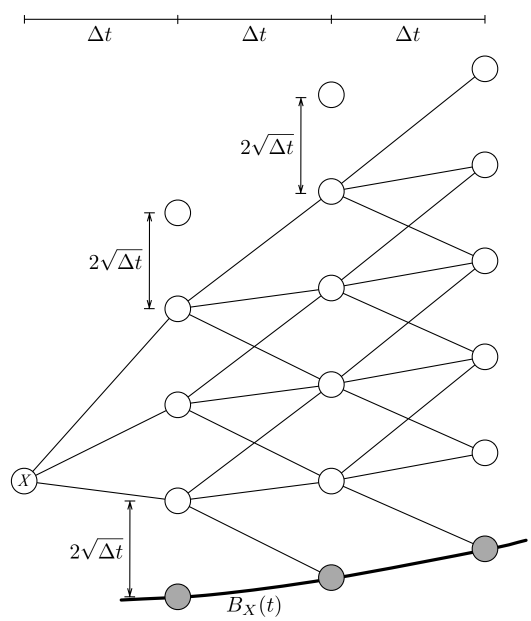

Figure 4: **The $X(t)$-Lattice with the Time-Varying Barrier $B_X(t)$.** The bold curve denotes the time-varying barrier $B_X(t)$. The log-distance between any two vertically adjacent nodes is $2\sqrt{\Delta t}$. The option knocks out when the gray nodes are hit.

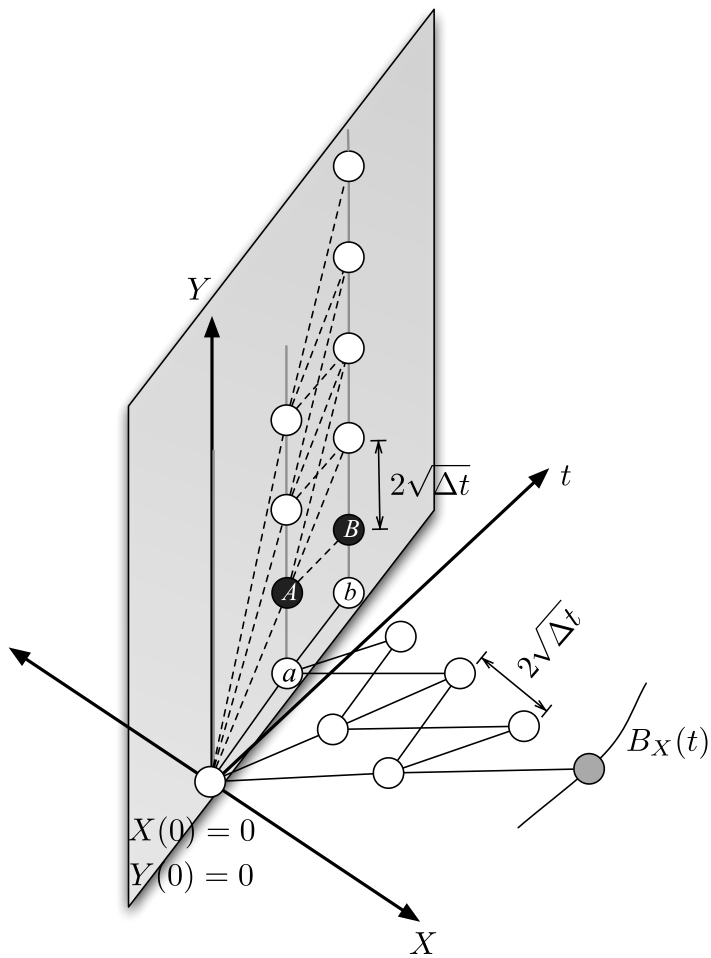

Figure 5: **Construction for the $Y(t)$-Lattice.** The curve denotes the time-varying barrier $B_X(t)$ on the X-t plane. The log-distances between any two vertically adjacent nodes on the X-t and Y-t planes are both $2\sqrt{\Delta t}$. The option knocks out when $X(t)$ reaches the gray nodes. The firm defaults when $Y(t)$ reaches the black nodes.

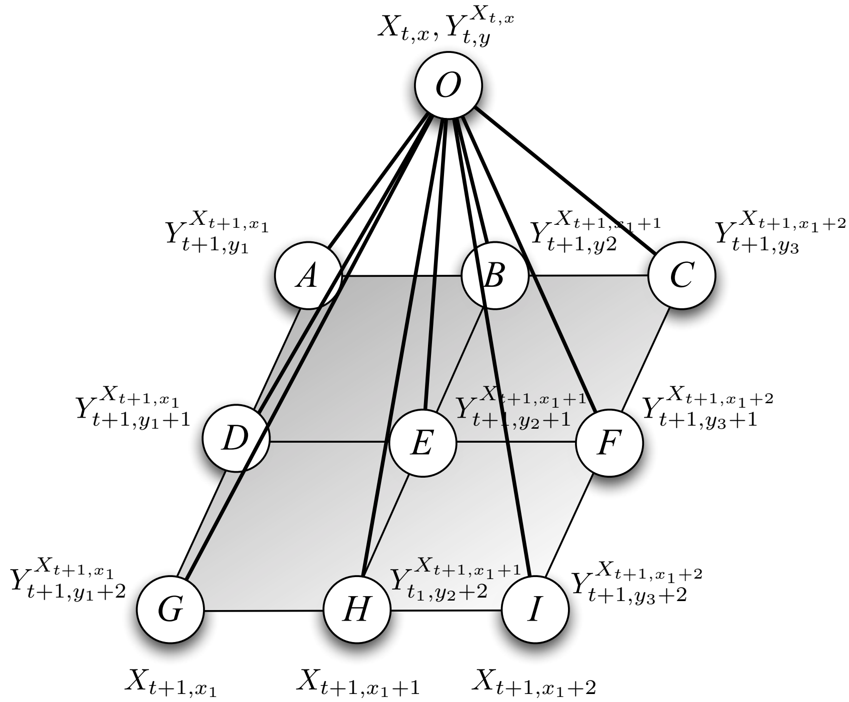

Figure 6: **The** $(X(t), Y(t))$ **Bivariate Lattice.** Node O at time step t has 9 branches leading to nodes A, B, C, D, E, F, G, H, and I at time step $t+1$. Let $X_{i,j}$ denote the jth highest value on the $X(t)$-lattice at time step i, and $Y_{i,j}^{k}$ denote the jth highest value on the $Y(t)$-lattice associated with the value k on the $X(t)$-lattice at time step i. The value on the $X(t)$-lattice at nodes A, D, and G is X_{t+1,x_1}, that at nodes B, E, and H is X_{t+1,x_1+1}, and that at nodes C, F, and I is X_{t+1,x_1+2}. The values on the $Y(t)$-lattices are listed next to the nodes.

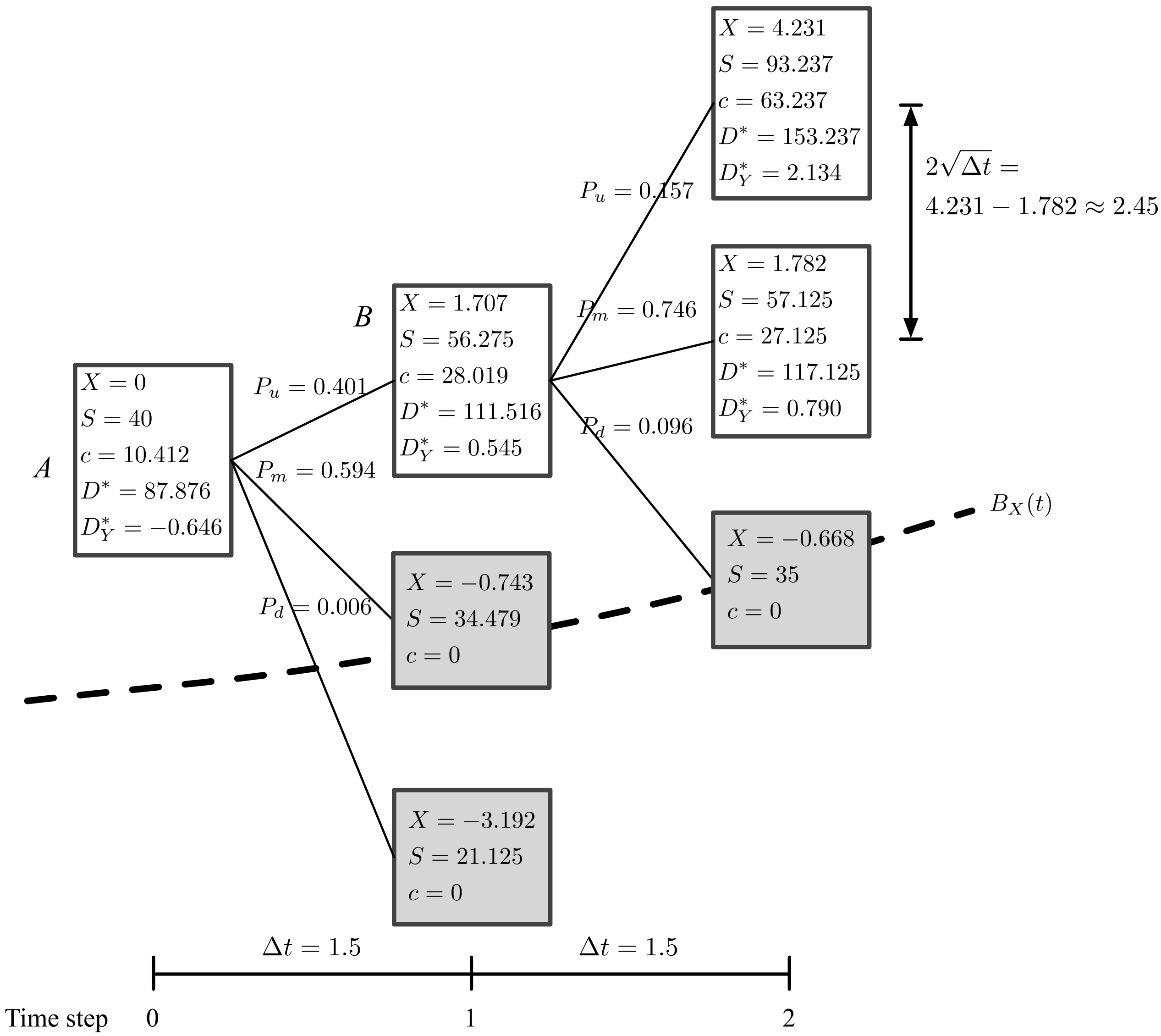

Figure 7: **A 2-Time-Step $X(t)$-Lattice.** The dashed curve denotes the time-varying barrier $B_X(t)$. The log-distance between any two vertically adjacent nodes is $2\sqrt{\Delta t} = 2.45$. The option knocks out when $X(t)$ reaches the gray nodes.

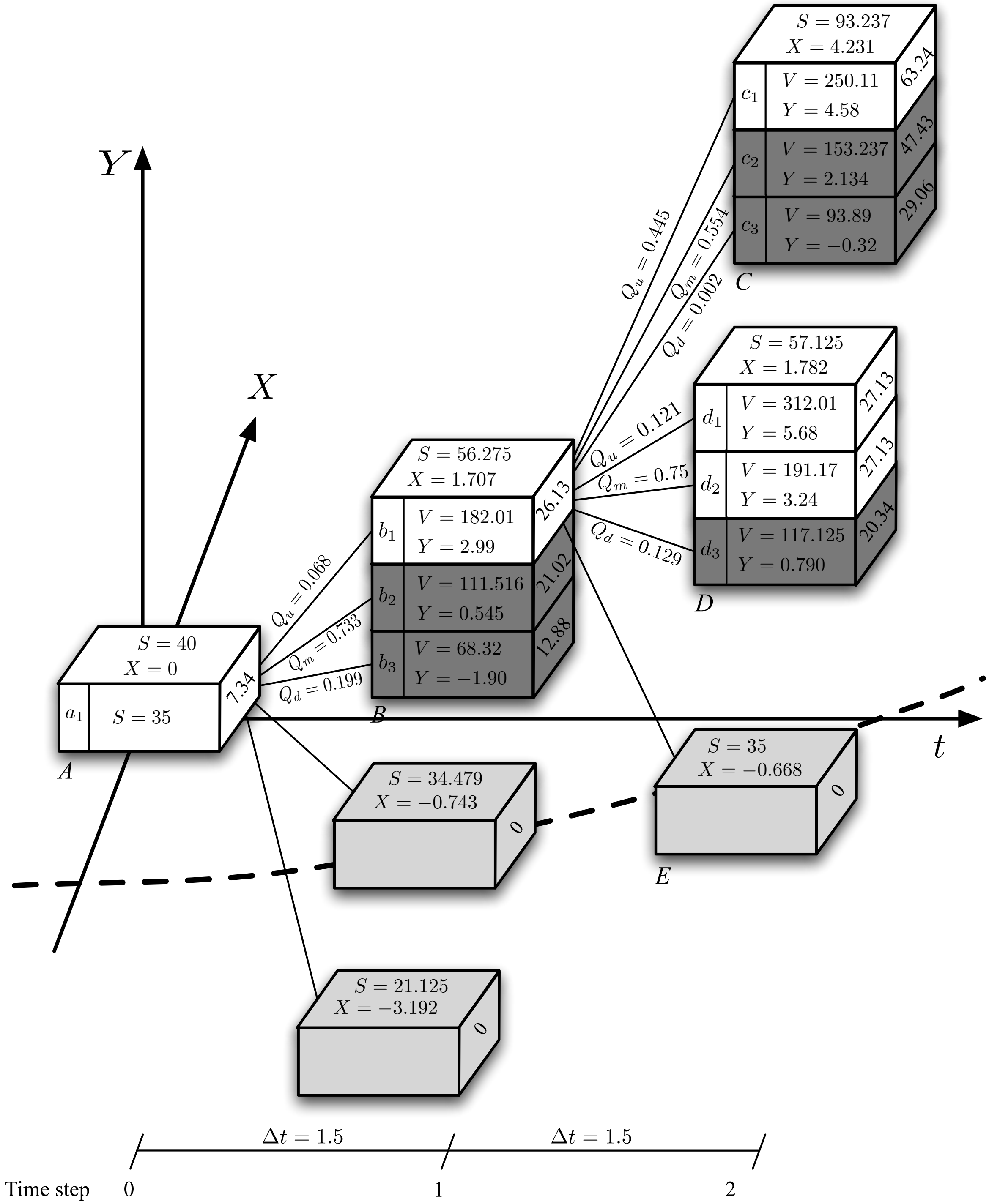

Figure 8: **A 2-Time-Step Bivariate Lattice.** The curve denotes the time-varying barrier $B_X(t)$ on the X-t plane. The log-distances between any two vertically adjacent nodes on the X-t and Y-t planes are both $2\sqrt{\Delta t}$. The option knocks out when $X(t)$ reaches the light-gray nodes. The firm defaults when $Y(t)$ reaches the dark-gray nodes.

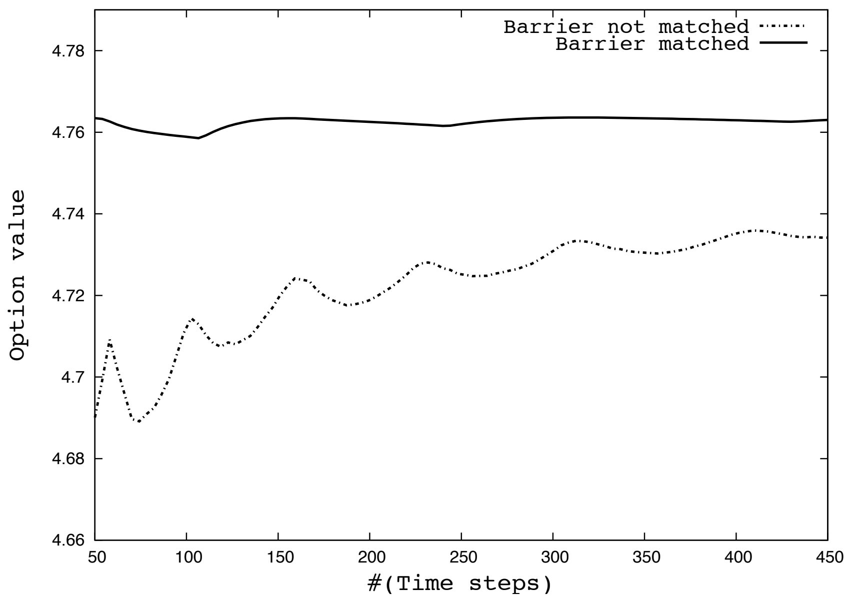

Figure 9: **Convergence of the Vulnerable Barrier Call Option under the FPM Model.** The x and the y axes denote the number of time steps and the option value, respectively. A barrier call option is assumed to be issued by a firm at time 0 and the call matures at $T = 3$ (years). The amount of all other outstanding debts of the firm is $D = 90$. The initial stock price is $S(0) = 40$, the strike price is $K = 40$, the initial firm's asset value is $V(0) = 100$, the volatility of the stock price is $\sigma_S = 0.2$, the volatility of the firm's asset value is $\sigma_V = 0.2$, the correlation between the stock price and firm's asset value is $\rho = 0$, the risk-free interest rate is $r = 0.05$, and the recovery rate is $\alpha = 0.75$, The barrier is 35. The default boundary for the firm's asset value at time t is $D^*(t) = De^{-r(T-t)} + c(S(t), t)$.

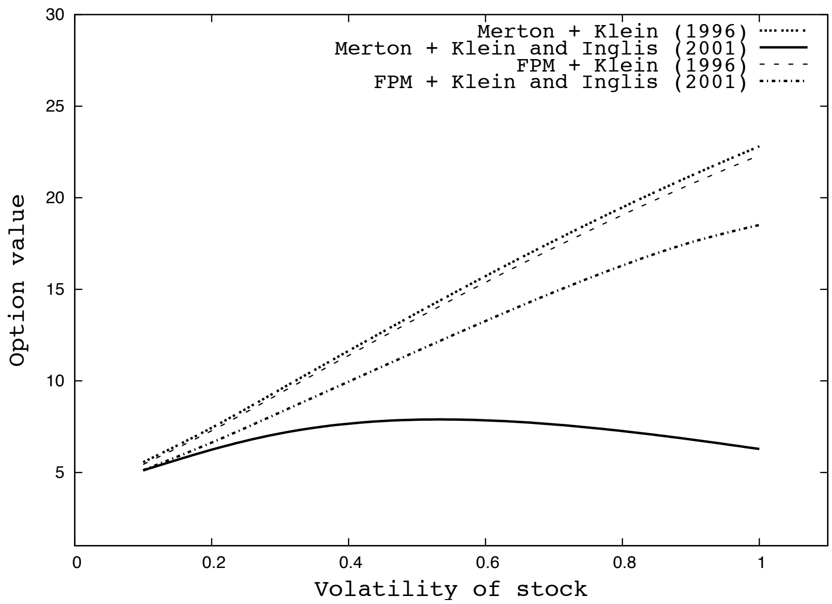

Figure 10: **Vanilla Option Values with Various Stock Price Volatilities.** A call option is assumed to be issued by a firm at time 0 and the call matures at $T = 3$ (years). The amount of all other outstanding debts of the firm is $D = 90$. The initial stock price is $S(0) = 40$, the strike price is $K = 40$, the initial firm's asset value is $V(0) = 100$, the volatility of the stock price is $\sigma_S = 0.2$, the volatility of the firm's asset value is $\sigma_V = 0.2$, the correlation between the stock price and firm's asset value is $\rho = 0$, the risk-free interest rate is $r = 0.05$, the recovery rate is $\alpha = 0.75$, and the number of the time steps is $n = 500$. In Klein (1996), the default boundary for the firm's asset value at time t is $D^*(t) = De^{-r(T-t)}$, whereas in Klein and Inglis (2001), the default boundary for the firm's asset value at time t is $D^*(t) = De^{-r(T-t)} + c(S(t), t)$, where $c(S(t), t)$ denotes the value of the call option at time t. In Merton's model, the default can only occur at maturity, whereas the default can occur prior to maturity in the FPM.

	Merton				FPM	
	D		$D + c(S(T), T)$		$De^{-r(T-t)}$	$De^{-r(T-t)} + c(S(t), t)$
	`Formula`	`Lattice`	`Formula`	`Lattice`	`Lattice`	`Lattice`
Base case	7.44	7.41	6.24	6.22	7.28	6.63
$S = 30$	2.27	2.26	2.01	2.01	2.22	2.09
$S = 50$	14.75	14.69	11.59	11.58	14.43	12.70
$V = 90$	7.03	7.00	5.71	5.70	6.87	6.37
$V = 110$	7.74	7.71	6.65	6.67	7.60	6.89
$\rho = 0.5$	8.06	8.04	7.36	7.33	7.71	7.07
$\rho = -0.5$	6.59	6.56	5.23	5.23	6.84	6.41
$\sigma_S = 0.15$	6.45	6.42	5.66	5.66	6.31	5.84
$\sigma_S = 0.25$	8.48	8.44	6.70	6.71	8.29	7.45
$\sigma_V = 0.15$	7.80	7.79	6.47	6.45	7.60	6.70
$\sigma_V = 0.25$	7.10	7.09	5.98	5.98	7.06	6.57
$T = 2$	5.79	5.77	4.97	4.98	5.62	5.14
$T = 4$	8.91	8.89	7.27	7.28	8.79	7.97
$\alpha = 1$	7.93	7.93	7.11	7.13	8.37	8.37
$\alpha = 0.5$	6.95	6.89	5.36	5.32	6.19	4.89
$r = 0.03$	6.17	6.14	5.16	5.14	6.03	5.58
$r = 0.07$	8.80	8.77	7.40	7.39	8.62	7.76

Table 1: **Pricing Vulnerable Vanilla Call Options.** A call option is assumed to be issued by a firm at time 0 and the call matures at $T = 3$ (years). The amount of all other outstanding debts of the firm is $D = 90$. For the base case, the initial stock price is $S(0) = 40$, the strike price is $K = 40$, the initial firm's asset value is $V(0) = 100$, the volatility of the stock price is $\sigma_S = 0.2$, the volatility of the firm's asset value is $\sigma_V = 0.2$, the correlation between the stock price and firm's asset value is $\rho = 0$, the risk-free interest rate is $r = 0.05$, and the recovery rate is $\alpha = 0.75$. The numerical settings of the other cases are the same as those of the base case except the parameter in the first column. The columns under `Merton` assume that the firm can only default at the option's maturity date, and the debt obligation at maturity is listed in the second row. The columns under `FPM` assume that the firm is allowed to default prior to maturity, and the default boundary at time t for $t \in [0, T]$ is listed in the second row. The prices under `Formula` are generated by the analytical formulas in Klein (1996) (the second column) and Klein and Inglis (2001) (the fourth column), and the prices under `Lattice` are generated by our lattice with 500 time steps.

		Merton			FPM	
		D		$D + c(S(T), T)$	$De^{-r(T-t)}$	$De^{-r(T-t)} + c(S(t), t)$
	B	`Formula`	`Lattice`	`Lattice`	`Lattice`	`Lattice`
	20	7.44	7.41	6.22	7.28	6.63
	25	7.43	7.40	6.21	7.27	6.62
Constant barrier	30	7.15	7.12	5.96	6.99	6.36
	35	5.39	5.37	4.41	5.27	4.76
	40	0.00	0.00	0.00	0.00	0.00
	20	7.44	7.41	6.22	7.28	6.63
	25	7.44	7.41	6.22	7.28	6.63
Exponential barrier	30	7.38	7.35	6.17	7.22	6.58
($\gamma = 0.06$)	35	6.93	6.90	5.75	6.78	6.16
	40	5.37	5.35	4.38	5.25	4.75

Table 2: **Pricing Vulnerable Barrier Options under Different Structural Models and Barrier Settings.** A barrier call option is assumed to be issued by a firm at time 0 and the call matures at $T = 3$ (years). The amount of all other outstanding debts of the firm is $D = 90$. The initial stock price is $S(0) = 40$, the strike price is $K = 40$, the initial firm's asset value is $V(0) = 100$, the volatility of the stock price is $\sigma_S = 0.2$, the volatility of the firm's asset value is $\sigma_V = 0.2$, the correlation between the stock price and firm's asset value is $\rho = 0$, the risk-free interest rate is $r = 0.05$, and the recovery rate is $\alpha = 0.75$. The barrier B for the constant barrier option is listed in the second column, and the barrier for the exponential barrier option at time t is equal to $B(t) = Be^{-\gamma(T-t)}$. The columns under `Merton` assume that the firm can only default at the option's maturity date, and the debt obligation at maturity is listed in the second row. The columns under `FPM` assume that the firm is allowed to default prior to maturity, and the default boundary at time t for $t \in [0, T]$ is listed in the second row. The prices under `Formula` are generated by the analytical formulas in Pan (2010), and the prices under `Lattice` are generated by our lattice with 500 time steps.

	Defaultable Zero Bonds with an Exogenous Default Boundary			
Face value (F)	`Lattice`	`Formula`	Relative errors	Default-free bonds
2000	1924.8	1924.8	-0.002%	1925.0
2500	2404.0	2404.4	-0.015%	2406.2
3000	2874.9	2876.2	-0.045%	2887.4

Table 3: **Pricing Defaultable Zero-Coupon Bonds with an Exogenous Default Boundary.** The initial firm's asset value is $V(0) = 5000$, the volatility of the firm's asset is $\sigma_V = 30\%$, the maturity is $T = 1$, the correlation between the firm's asset value and the interest rate is $\rho = -0.25$, the recovery rate is $\alpha = 1$, and the tax rate is $\tau = 0$. The parameters for the interest rate model are $a = 50\%$ and $\sigma_r = 1\%$. The default boundary at time t is $\xi = 90\%$ of the present value of the bond's face value. The annualized yields of zero-coupon bonds for years 0.5 and 1 are 0.0343 and 0.0382, respectively, and those for any other time $t \in [0, 1]$ is calculated by linear interpolation. `Lattice` and `Formula` denote the option values computed by our lattice with 180 time steps and by the analytical formula proposed by Briys and De Varenne (1997). The relative errors are defined as $(\texttt{Lattice} - \texttt{Formula})/\texttt{Formula}$.

Payment frequency	Total-asset-sales		No-asset-sales		
(per year)	$\xi = 0.9$	$\xi = 0$	$\xi = 0.9$	$\xi = 0$	Default-free bonds
Continuously	3017.6	3014.6	3021.4	3019.9	3034.8
4	3016.5	3013.4	3019.7	3017.7	3034.1
2	3015.3	3012.2	3017.7	3015.3	3033.4
1	3012.8	3009.5	3012.8	3009.5	3031.8

Table 4: **Pricing Coupon Bonds under Different Payment Frequencies, Asset Sales Assumptions, and Default Boundaries.** The initial firm's asset value is $V(0) = 5000$, the volatility of the firm's asset is $\sigma_V = 30\%$, the face value of the outstanding bond is $F = 3000$, the annualized coupon rate is $c = 5\%$, the maturity is $T = 1$, the correlation between the firm's asset value and the interest rate is $\rho = -0.25$, the recovery rate is $\alpha = 1$, and the tax rate is $\tau = 0$. The parameters for the interest rate model are $a = 50\%$ and $\sigma_r = 1\%$. The default boundary at time t is set as ξ of the sum of the present values of the unpaid coupons and the face value at maturity of the defaultable bond. The annualized yields of zero-coupon bonds for years 0.5 and 1 are 0.0343 and 0.0382, respectively, and those for any other time $t \in [0, 1]$ is calculated by linear interpolation. The bond prices are evaluated numerically by our lattice with 180 time steps.

Call/ Put	Callable bonds		Putable bonds		Straight bonds	
prices	Defaultable	Default-free	Defaultable	Default-free	Defaultable	Default-free
3030	3016.6	3029.4	3063.6	3063.8	3017.6	3034.8
3035	3017.2	3033.0	3068.4	3068.7	3017.6	3034.8
3040	3017.5	3034.4	3073.2	3073.5	3017.6	3034.8
3050	3017.6	3034.8	3082.9	3083.2	3017.6	3034.8

Table 5: **Pricing Callable and Putable Bonds.** The initial firm's asset value is $V(0) = 5000$, the volatility of the firm's asset is $\sigma_V = 30\%$, the face value is $F = 3000$, the annualized coupon rate is $c = 5\%$, the maturity is $T = 1$, the correlation between the firm's asset value and the interest rate is $\rho = -0.25$, the recovery rate is $\alpha = 1$, and the tax rate is $\tau = 0$. The parameters for the interest rate model are $a = 50\%$ and $\sigma_r = 1\%$. The annualized yields of zero-coupon bonds for years 0.5 and 1 are 0.0343 and 0.0382, respectively, and those for any other time $t \in [0, 1]$ is calculated by linear interpolation. The default boundary for defaultable bonds at time t is $\xi = 90\%$ of the sum of the present values of the unpaid coupons and the face value at maturity. The continuous coupon payments are fully financed by selling the firm's asset. The bond prices are evaluated numerically by our lattice with 180 time steps.

	Charges (bps)		
T	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$
1	140.43	141.85	142.48
5	42.94	44.21	45.32
10	19.07	20.13	21.15
20	7.73	8.48	9.21
25	5.61	6.25	6.86

Table 6: **Charges Change with Maturities and Correlations.** The initial account value is $W(0) = 100$, and the volatility of the account value is $\sigma_W = 20\%$. In the table, T represents the maturity, and ρ represents the correlation between the account value and the interest rate. The number of the time steps is $n = 20 \times T$, and the annual withdrawal is $G = W(0)/T$. The curve of annualized yields of zero-coupon bonds is generated by the Vasicek model with parameters $r(0) = 3\%$, $\theta = 0.3\%$, $\sigma_r = 1\%$, and $a = 0.1$.