Stochastic Calculus for Finance II Continuous-Time Models
Chapter 3 Exercise

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√ **Exercise 3.3 (Normal kurtosis).** The *kurtosis* of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. This fact was used to obtain (3.4.7). This exercise verifies this fact.

Let X be a normal random variable with mean μ, so that $X-\mu$ has mean zero. Let the variance of X, which is also the variance of $X-\mu$, be σ^2. In (3.2.13), we computed the moment-generating function of $X-\mu$ to be $\varphi(u) = \mathbb{E}e^{u(X-\mu)} = e^{\frac{1}{2}u^2\sigma^2}$, where u is a real variable. Differentiating this function with respect to u, we obtain

$$\varphi'(u) = \mathbb{E}\left[(X-\mu)e^{u(X-\mu)}\right] = \sigma^2 u e^{\frac{1}{2}\sigma^2 u^2}$$

and, in particular, $\varphi'(0) = \mathbb{E}(X-\mu) = 0$. Differentiating again, we obtain

$$\varphi''(u) = \mathbb{E}\left[(X-\mu)^2 e^{u(X-\mu)}\right] = \left(\sigma^2 + \sigma^4 u^2\right) e^{\frac{1}{2}\sigma^2 u^2}$$

and, in particular, $\varphi''(0) = \mathbb{E}\left[(X-\mu)^2\right] = \sigma^2$. Differentiate two more times and obtain the normal kurtosis formula $\mathbb{E}\left[(X-\mu)^4\right] = 3\sigma^4$.

Ans.

$$\varphi'''(u) = \mathrm{E}[(X-\mu)^3 e^{u(X-\mu)}] = (3\sigma^4 u + \sigma^6 u^3)e^{\frac{1}{2}\sigma^2 u^2}$$

$$\varphi''''(u) = \mathrm{E}[(X-\mu)^4 e^{u(X-\mu)}] = (3\sigma^4 + 3\sigma^6 u^2)e^{\frac{1}{2}\sigma^2 u^2} + \sigma^2 u(3\sigma^4 u + \sigma^6 u^3)e^{\frac{1}{2}\sigma^2 u^2}$$

$$\therefore \varphi''''(0) = \mathrm{E}[(X-\mu)^4 e^{0(X-\mu)}] = \mathrm{E}[(X-\mu)^4] = 3\sigma^4$$

Exercise 3.6 Let $W(t)$ be a Brownian motion and let $F(t), t \geq 0$, be an associated filtration.

(i) For $\mu \in \mathbb{R}$, consider the *Brownian motion with drift* μ:

$$X(t) = \mu t + W(t).$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \leq s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y-x-\mu(t-s))^2}{2(t-s)}\right\} dy$$

satisfies $\mathbb{E}\big[f\big(X(t)\big)\big|\mathcal{F}(s)\big] = g\big(X(s)\big)$, and hence X has the Markov property. We may rewrite $g(x)$ as $g(x) = \int_{-\infty}^{\infty} f(y)p(\tau, x, y)\, dy$, where $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(y-x-\mu\tau)^2}{2\tau}\right\}$$

is the *transition density* for Brownian motion with drift μ.

(ii) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the *geometric Brownian motion*

$$S(t) = S(0)e^{\sigma W(t) + \nu t}.$$

Set $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sigma y\sqrt{2\pi\tau}} \exp\left\{-\frac{\left(\log \frac{y}{x} - \nu\tau\right)^2}{2\sigma^2\tau}\right\}.$$

Show that for any Borel-measurable function $f(y)$ and for any $0 \leq s < t$ the function $g(x) = \int_0^{\infty} h(y)p(\tau, x, y)\, dy$ satisfies $\mathbb{E}\big[f\big(S(t)\big)\big|\mathcal{F}(s)\big] = g\big(S(s)\big)$ and hence S has the Markov property and $p(\tau, x, y)$ is its transition density.

Ans.

(i)

$\mathrm{E}[f(X(t)) \mid F_s] = E[f(\mu t + W(t)) \mid F_s]$

$= E[f(\mu(t-s) + W(t) - W(s) + X(s)) \mid F_s]$

$\because \mu(t-s) + W(t) - W(s) \sim N(\mu(t-s), t-s)$

Let $y = \mu(t-s) + W(t) - W(s) + X(s)$

$y - X(s) \sim N(\mu(t-s), t-s)$

$\therefore \mathrm{E}[f(X(t)) \mid F_s] = \int_{-\infty}^{\infty} f(y)p(\tau = t-s, X(s), y)dy$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\{-\frac{(y - X(s) - \mu(t-s))^2}{2(t-s)}\}dy$$

$= g(X(s))$

$\therefore X$ has the Markov property.

(ii)

$$\because \log(\frac{S(t)}{S(s)}) = \sigma(W(t) - W(s)) + \nu(t-s), W(t) - W(s) \sim N(0, t-s)$$

$$\therefore \log(\frac{S(t)}{S(s)}) \sim N(\nu\tau, \sigma\tau), \tau = t - s$$

Let $y = S(t), z = \log(\frac{y}{S(s)}) \Rightarrow \log(\frac{y}{S(s)}) \sim N(\nu\tau, \tau), dz = \frac{1}{y}dy$

$$\therefore \mathrm{E}[f(S(t)) \mid F_s] = \int_{-\infty}^{\infty} f(y)p(\tau, S(s), y)\frac{1}{y}dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma y\sqrt{2\pi\tau}} f(y)\exp\{-\frac{(\log(\frac{y}{S(s)}) - \nu\tau)^2}{2\sigma^2\tau}\}dy = g(S(s)) \text{ where } \tau = t - s$$

$\therefore S$ has the Markov property.

補充

Exercise1.

Let $c > 0$ be a constant. Prove that

(1) $X_t := W_{t+c} - W_c$ is a Brownian motion.

(2) $X_t := \frac{1}{c} W_{c^2 t}$ is a Brownian motion.

Ans.

(1)

$1. X_0 = W_{0+c} - W_c = 0$

$2. X_{t+s} - X_t = (W_{t+s+c} - W_c) - (W_{t+c} - W_c) = W_{t+s+c} - W_{t+c} \sim N(0,s)$

$3. \forall t_1 \le t_2 \le t_3 \le t_4$

$X_{t_4} - X_{t_3} = W_{t_4+c} - W_{t_3+c}, X_{t_2} - X_{t_1} = W_{t_2+c} - W_{t_1+c}$

$\because t_1 + c \le t_2 + c \le t_3 + c \le t_4 + c, W_t$ is a B.M

$\therefore W_{t_4+c} - W_{t_3+c} \perp W_{t_2+c} - W_{t_1+c} \Rightarrow X_{t_4} - X_{t_3} \perp X_{t_2} - X_{t_1}$

By 1.2.3 and definition 3.3.1, X_t is a B.M.

(2)

$1. X_0 = \frac{1}{c} W_{c^2 \bullet 0} = \frac{1}{c} W_0 = 0$

$2. X_{t+s} - X_t = \frac{1}{c}(W_{c^2(t+s)} - W_{c^2 t})$

$\because (W_{c^2(t+s)} - W_{c^2 t}) \sim N(0, c^2(t+s) - c^2(t)) = N(0, c^2 s) \therefore X_{t+s} - X_t \sim N(0,s)$

$3. \forall t_1 \le t_2 \le t_3 \le t_4$

$X_{t_4} - X_{t_3} = \frac{1}{c}(W_{c^2 t_4} - W_{c^2 t_3}), X_{t_2} - X_{t_1} = \frac{1}{c}(W_{c^2 t_2} - W_{c^2 t_1})$

$\because c^2 t_1 \le c^2 t_2 \le c^2 t_3 \le c^2 t_4, W_t$ is a B.M

$\therefore (W_{c^2 t_4} - W_{c^2 t_3}) \perp (W_{c^2 t_2} - W_{c^2 t_1})$

$$\forall x, y \in \mathbb{R}, P(X_{t_4} - X_{t_3} \le x; X_{t_2} - X_{t_1} \le y) = P(\frac{1}{c}(W_{c^2 t_4} - W_{c^2 t_3}) \le x; \frac{1}{c}(W_{c^2 t_2} - W_{c^2 t_1}) \le y)$$

$$= P((W_{c^2 t_4} - W_{c^2 t_3}) \le cx; (W_{c^2 t_2} - W_{c^2 t_1}) \le cy)$$

$$= P((W_{c^2 t_4} - W_{c^2 t_3}) \le cx) P((W_{c^2 t_2} - W_{c^2 t_1}) \le cy) \quad \because (W_{c^2 t_4} - W_{c^2 t_3}) \perp (W_{c^2 t_2} - W_{c^2 t_1})$$

$$= P(\frac{1}{c}(W_{c^2 t_4} - W_{c^2 t_3}) \le x) P(\frac{1}{c}(W_{c^2 t_2} - W_{c^2 t_1}) \le y)$$

$$= P(X_{t_4} - X_{t_3} \le x) P(X_{t_2} - X_{t_1} \le y)$$

$\therefore X_{t_4} - X_{t_3} \perp X_{t_2} - X_{t_1}$

By 1.2.3 and definition 3.3.1, X_t is a B.M.

Exercise2.

Check whether the following processes are maringales.

(1) $X_t = W_t + 4t$.

(2) $X_t = W_t^2$.

(3) $X_t = W_t^3 - 3tW_t$.

Ans.

(1)

$\mathrm{E}[X_t \mid F_s] = \mathrm{E}[W_t + 4t \mid F_s], t > s$

$= \mathrm{E}[(W_t - W_s) + 4(t-s) + X_s \mid F_s]$

$\because (W_t - W_s)$ and $(t-s)$ are independent of F_s

$= \mathrm{E}[(W_t - W_s)] + 4(t-s) + X_s$

$= 4(t-s) + X_s > X_s$

$\therefore X_t$ is not a martingale.

(2)

$\mathrm{E}[X_t \mid F_s] = \mathrm{E}[(W_s + (W_t - W_s))^2 \mid F_s], t > s$

$= \mathrm{E}[W^2{}_s + 2W_s(W_t - W_s) + (W_t - W_s)^2 \mid F_s]$

$\because (W_t - W_s)^2$ and $(W_t - W_s)$ are independent of F_s

$= X_s + 2W_s\mathrm{E}[W_t - W_s] + \mathrm{E}[(W_t - W_s)^2]$

$(\because (W_t - W_s) \sim N(0, t-s) \therefore \mathrm{E}[(W_t - W_s)^2] = \mathrm{var}[(W_t - W_s)] = t - s)$

$= (t+s) + X_s > X_s$

$\therefore X_t$ is not a martingale.

(3)

$\mathrm{E}[X_t \mid F_s] = \mathrm{E}[((W_t - W_s) + W_s)^3 - 3(t - s + s)((W_t - W_s) + W_s) \mid F_s], t > s$

Let $A = (W_t - W_s), \tau = t - s$

$\mathrm{E}[X_t \mid F_s] = \mathrm{E}[A^3 + 3A^2W_s + 3A(W_s{}^2) - 3\tau A - 3\tau W_s - 3sA + W_s{}^3 - 3sW_s \mid F_s]$

$\because \tau, A^3, A^2$ and A are independent of F_s

$= \mathrm{E}[A^3] + 3\mathrm{E}[A^2]W_s + 3\mathrm{E}[A](W_s{}^2 - \tau - s) - 3\tau W_s + X_s = X_s$

$\therefore X_t$ is a martingale.

Hint: $A \sim N(0, t-s), \phi(u) = E[e^{uA}] = e^{\frac{1}{2}u^2(t-s)^2}, \phi'''(0) = E[A^3] = 0$