# Complexity Analysis



# Complexity

- Space
  - The amount of memory space needed to run the program.
- Time
  - The amount of computational time needed to run the program

We use insertion sort as an example

Pick an instance characteristic … n

n = a.length (the number of elements to be sorted)



# Space Complexity for Insertion Sort

```
for (int i = 1; i < a.length; i++)
{// insert a[i] into a[0:i-1]
  int t = a[i];
  int j;
  for (j = i - 1; j >= 0 && t < a[j];
   j--)
     a[j + 1] = a[j];
  a[j + 1] = t;
}
```

Fixed part:
 independent of n
 ex: instruction space
 Variables: i, j,,t
Variable part:
 size dependent on n
 ex: a[]

Space requirement=
Fixed + Variable

Focus on variable part:
a[] → n



# Time Complexity

- Count a particular operation
- Count number of steps
- Asymptotic complexity

## Comparison Count

```
for (int i = 1; i < a.length; i++)
{// insert a[i] into a[0:i-1]
  int t = a[i];
  int j;
  for (j = i - 1; j >= 0 && t < a[j]; j--)
     a[j + 1] = a[j];
  a[j + 1] = t;
}
```

- Determine the number of comparison count as a function of n



## Comparison Count

```
for (j = i - 1; j >= 0 && t < a[j]; j--)
     a[j + 1] = a[j];
```

How many comparisons are made?
Number of compares depends on
a[], t and i



## Comparison Count

- Worst-case count = maximum count
- Best-case count = minimum count
- Average count



## Worst-Case Comparison Count

```
for (j = i - 1; j >= 0 && t < a[j]; j--)
     a[j + 1] = a[j];
```

a = [1, 2, 3, 4] and t = 0 => 4 compares
a = [1,2,3,…,n] and t = 0 => n compares

## Worst-Case Comparison Count

```
for (int i = 1; i < n; i++)
   for (j = i - 1; j >= 0 && t < a[j]; j--)
       a[j + 1] = a[j];
```

total compares = 1 + 2 + 3 + … + (n-1)

= (n-1)n/2



## In Class Exercise: Best Case Comparison Count

```
for (int i = 1; i < n; i++)
   for (j = i - 1; j >= 0 && t < a[j]; j--)
       a[j + 1] = a[j];
```

- a = [1, 2, 3, 4] and t = 5 => 1 compares
- a = [1,2,3,…,n] and t = n+1 =>1 compares
- Compute the total number of comparison



## Step Count

A step is an amount of computing that does not depend on the instance characteristic n

10 adds, 100 subtracts, 1000 multiplies can all be counted as a single step

n adds cannot be counted as 1 step



## Step per execution (s/e)

| | s/e |
|---|---|
| for (int i = 1; i < a.length; i++) | 1 |
| {// insert a[i] into a[0:i-1] | 0 |
| int t = a[i]; | 1 |
| int j; | 0 |
| for (j = i - 1; j >= 0 && t < a[j]; j--) | 1 |
| a[j + 1] = a[j]; | 1 |
| a[j + 1] = t; | 1 |
| } | 0 |

## Step per execution

s/e isn't always 0 or 1

x = sum(a, n);

where n is the instance characteristic and
sum adds a[0:n-1] has a s/e count of n
(a[0]+a[1]+a[2]+…+a[n-1])



## Step Count

| | s/e | steps |
|---|---|---|
| for (int i = 1; i < a.length; i++) | 1 | |
| {// insert a[i] into a[0:i-1] | 0 | |
| int t = a[i]; | 1 | |
| int j; | 0 | |
| for (j = i - 1; j >= 0 && t < a[j]; j--) | 1 | i+ 1 |
| a[j + 1] = a[j]; | 1 | i |
| a[j + 1] = t; | 1 | |
| } | 0 | |

Worst case analysis



## Step Count

for (int i = 1; i < a.length; i++)
{ 2i + 3}

step count for
for (int i = 1; i < a.length; i++)
is n

step count for body of for loop is
2(1+2+3+…+n-1) + 3(n-1)
= (n-1)n + 3(n-1)
= (n-1)(n+3)



| | s/e | frequency | total steps |
|---|---|---|---|
| for (int i = 1; i < a.length; i++) | 1 | n | n |
| { // insert a[i] into a[0:i-1] | 0 | n-1 | 0 |
| int t = a[i]; | 1 | n-1 | n-1 |
| int j; | 0 | n-1 | 0 |
| for (j = i - 1; j >= 0 && t < a[j];j--) | 1 | (n-1)(n+2)/2 | |
| a[j + 1] = a[j]; | 1 | n(n-1)/2 | |
| a[j + 1] = t; | 1 | n-1 | n-1 |
| } | 0 | n-1 | 0 |

Total：n^2+3n-3

# In Class Exercise:

Determine the s/e, **frequency counts, and total steps** for all statements in the following program segment

```
for(i=1;i<=n;i++)
      for(j=1;j<=i;j++)
         for(k=1;k<=j;k++)
            x++;
```



# Asymptotic Complexity of Insertion Sort

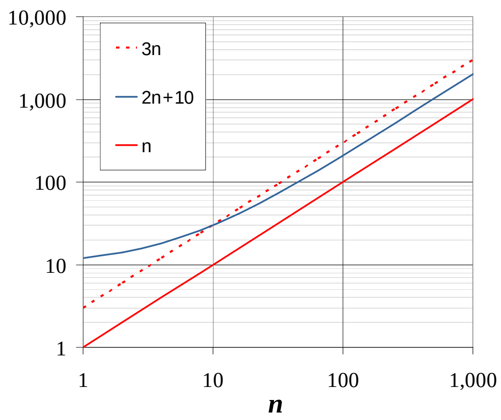
- (n-1)(n+3)→**O(n²)**
- What does this mean?



# Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants $c$ and $n_0$ such that

  $f(n) \le cg(n)$ for $n \ge n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $(c - 2)\, n \ge 10$
  - $n \ge 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



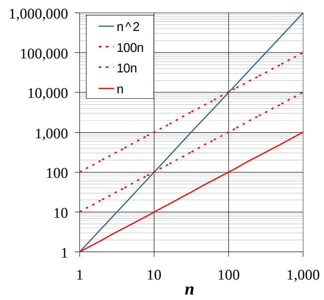



# Big-Oh Example

Example: the function $n^2$ is not $O(n)$

- $n^2 \le cn$
- $n \le c$

the above inequality cannot be satisfied since $c$ must be a constant



## Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

| | $f(n)$ is $O(g(n))$ | $g(n)$ is $O(f(n))$ |
|---|---|---|
| $g(n)$ grows more | Yes | No |
| $f(n)$ grows more | No | Yes |
| Same growth | Yes | Yes |



## Complexity of Insertion Sort

- Time or number of operations does not exceed $\mathbf{c.n^2}$ on any input of size $\mathbf{n}$ ($\mathbf{n}$ suitably large).
- Actually, the worst-case time is $\Theta(\mathbf{n^2})$ and the best-case is $\Theta(\mathbf{n})$
- So, the worst-case time is expected to quadruple each time $\mathbf{n}$ is doubled

The definition of $\Theta$ (n) will be discussed finally.

## Complexity of Insertion Sort

- Is $\mathbf{O(n^2)}$ too much time?
- Is the algorithm practical?

## Practical Complexities

$10^9$ instructions/second

| *n* | *n* | *nlogn* | $n^2$ | $n^3$ |
|---|---|---|---|---|
| *1000* | 1mic | 10mic | 1milli | 1sec |
| *10000* | 10mic | 130mic | 100milli | 17min |
| $10^6$ | 1milli | 20milli | 17min | 32years |

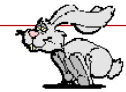
# Impractical Complexities

$10^9$ instructions/second

| *n* | $n^4$ | $n^{10}$ | $2^n$ |
|---|---|---|---|
| *1000* | 17min | $3.2 \times 10^{13}$ years | $3.2 \times 10^{283}$ years |
| *10000* | 116 days | ??? | ??? |
| $10^6$ | $3 \times 10^7$ years | ?????? | ?????? |



# Faster Computer v.s Better algorithm



Algorithmic improvement more useful than hardware improvement.

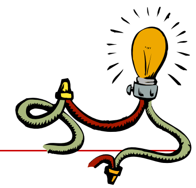
E.g. $2^n$ to $n^3$

# Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n)$ for $n \geq n_0$
- **little-oh**
  - f(n) is $o(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$
- **little-omega**
  - f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$



# Intuition for Asymptotic Notation



- **Big-Oh**
  - f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)
- **big-Omega**
  - f(n) is $\Omega(g(n))$ if f(n) is asymptotically **greater than or equal** to g(n)
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if f(n) is asymptotically **equal** to g(n)
- **little-oh**
  - f(n) is o(g(n)) if f(n) is asymptotically **strictly less** than g(n)
- **little-omega**
  - f(n) is $\omega(g(n))$ if f(n) is asymptotically **strictly greater** than g(n)

# Example

$$f(n) = 2n^2 + n + 4$$
$$g(n) = n^2$$
$$f(n) = \theta(g(n))$$

--------------

$$c' = 1, c'' = 7$$
$$1 * g(n) <= f(n) <= 7 * g(n),\ for\ n >= 1$$
$$1*1^2 <= 2*1^2 + 1 + 4 <= 7*1^2$$



# Example

$$f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0$$
$$f(n) = O(n^m)$$

$$\begin{aligned} f(n) &= a_m n^m + a_{m-1} n^{m-1} + \cdots a_1 n + a_0 \\ &= \sum_{i=0}^{m} a_i n^i \\ &\le \sum_{i=0}^{m} |a_i| n^i \\ &= n^m \cdot \sum_{i=0}^{m} |a_i| n^{i-m} \\ &\le n^m \cdot \sum_{i=0}^{m} |a_i| \end{aligned}$$



# Homework

Determine the frequency counts for all statements and analysis the complexity for the program segment

```
for(int i=0;i<n;i++)
  { // n is number of elements stored in array
   for (int j=0;j<n-i-1;j++)
       {
        if(array[j]>array[j+1])
          Swap(array[j],array[j+1]);
       }
}
```