# Single-Source All-Destinations Shortest Paths With General Weights

- Directed weighted graph.
- Edges may have negative cost.
- No cycle whose cost is $< 0$.
- Find a shortest path from a given source vertex s to each of the n vertices of the digraph.

# Single-Source All-Destinations Shortest Paths With General Weights

- Dijkstra's $O(n^2)$ single-source greedy algorithm doesn't work when there are negative-cost edges.

# Bellman-Ford Algorithm

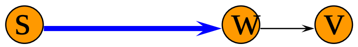
- Single-source all-destinations shortest paths in digraphs with negative-cost edges.
- Uses dynamic programming.
- Runs in $O(n^3)$ time when adjacency matrices are used.
- Runs in $O(ne)$ time when adjacency lists are used.

# Strategy


- To construct a shortest path from the source to vertex v, decide on the max number of edges on the path and on the vertex that comes just before v.

- Since the digraph has no cycle whose length is $< 0$, we may limit ourselves to the discovery of cycle-free (acyclic) shortest paths.
- A path that has no cycle has at most $n-1$ edges.

# Cost Function d




- Let d(v,k) (dist$^k$[v]) be the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most k edges.
- d(v,n-1) is the length of a shortest unconstrained path from the source vertex to vertex v.
- We want to determine d(v,n-1) for every vertex v.

# Value Of d(*,0)

- d(v,0) is the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most 0 edges.


- d(s,0) = 0.
- d(v,0) = infinity for v != s.

# Recurrence For d(*,k), k > 0

- d(v,k) is the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most k edges.
- If this constrained shortest path goes through no more than k-1 edges, then d(v,k) = d(v,k-1).

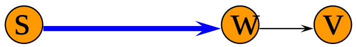
# Recurrence For d(*,k), k > 0

- If this constrained shortest path goes through k edges, then let w be the vertex just before v on this shortest path (note that w may be s).


- We see that the path from the source to w must be a shortest path from the source vertex to vertex w under the constraint that this path has at most k-1 edges.
- d(v,k) = d(w,k-1) + length of edge (w,v).

# Recurrence For d(*,k), k > 0

- d(v,k) = d(w,k-1) + length of edge (w,v).




- We do not know what w is.
- We can assert
  - d(v,k) = min{d(w,k-1) + length of edge (w,v)}, where the min is taken over all w such that (w,v) is an edge of the digraph.
- Combining the two cases considered yields:
  - d(v,k) = min{d(v,k-1),
    min{d(w,k-1) + length of edge (w,v)}}

# Pseudocode To Compute d(*,*)

```
// initialize d(*,0)
d(s,0) = 0;
d(v,0) = infinity, v != s;
// compute d(*,k), 0 < k < n
for (int k = 1; k < n; k++)
{
   d(v,k) = d(v,k-1), 1 <= v <= n;
   for (each edge (u,v))
      d(v,k) = min{d(v,k), d(u,k-1) + cost(u,v)}
}
```

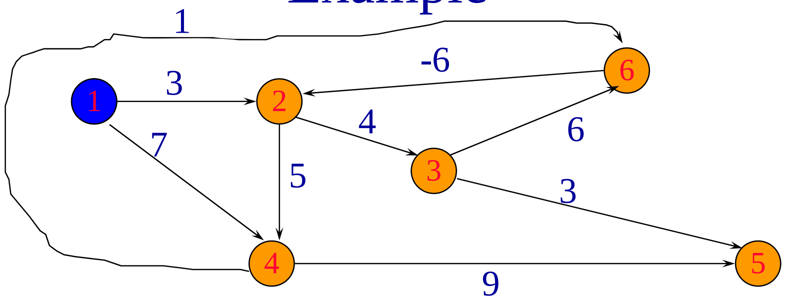
# Complexity

- $\Theta(n)$ to initialize d(*,0).
- $\Theta(n^2)$ to compute d(*,k) for each k > 0 when adjacency matrix is used.
- $\Theta(e)$ to compute d(*,k) for each k > 0 when adjacency lasts are used.
- Overall time is $\Theta(n^3)$ when adjacency matrix is used.
- Overall time is $\Theta(ne)$ when adjacency lists are used.
- $\Theta(n^2)$ space needed for d(*,*).

# p(*,*)

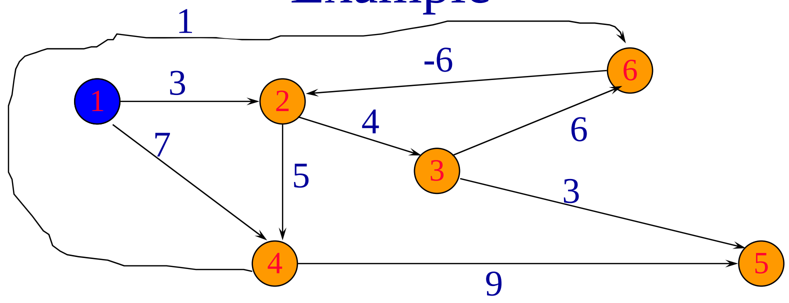
- Let p(v,k) be the vertex just before vertex v on the shortest path for d(v,k).
- p(v,0) is undefined.
- Used to construct shortest paths.

# Example




Source vertex is 1.

# Example




| k \ v | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 0 | - | - | - | - | - |
| 1 | 0 | 3 | - | 7 | - | - |
| 2 | 0 | 3 | 7 | 7 | 16 | 8 |
| 3 | 0 | 2 | 7 | 7 | 10 | 8 |
| 4 | 0 | 2 | 6 | 7 | 10 | 8 |

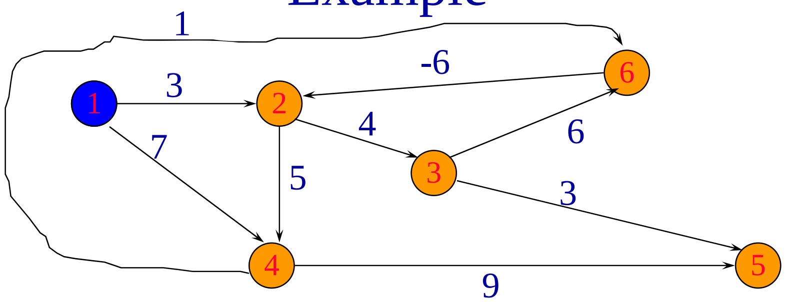
d(v,k)

v →, k ↓

| | | | | | |
|---|---|---|---|---|---|
| - | - | - | - | - | - |
| - | 1 | - | 1 | - | - |
| - | 1 | 2 | 1 | 4 | 4 |
| - | 6 | 2 | 1 | 3 | 4 |
| - | 6 | 2 | 1 | 3 | 4 |

p(v.k)

# Example




| k \ v | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 4 | 0 | 2 | 6 | 7 | 10 | 8 |
| 5 | 0 | 2 | 6 | 7 | 9 | 8 |

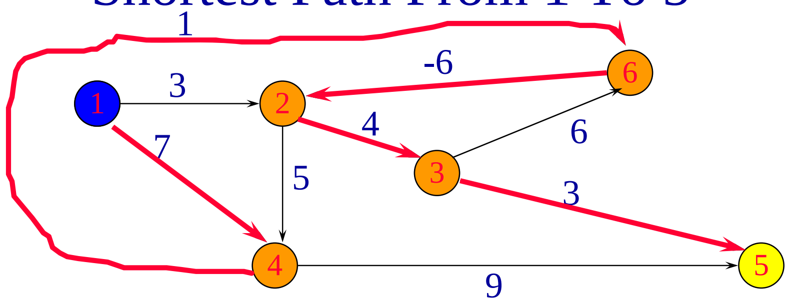
v →, k ↓

| | | | | | |
|---|---|---|---|---|---|
| - | 6 | 2 | 1 | 3 | 4 |
| - | 6 | 2 | 1 | 3 | 4 |

d(v,k)

p(v.k)

# Shortest Path From 1 To 5




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 5 | 0 | 2 | 6 | 7 | 9 | 8 |

d(v,5)

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| - | 6 | 2 | 1 | 3 | 4 |

p(v,5)

# Observations

- $d(v,k) = \min\{d(v,k-1),$
  $\min\{d(w,k-1) + \text{length of edge } (w,v)\}\}$
- $d(s,k) = 0$ for all $k$.
- If $d(v,k) = d(v,k-1)$ for all $v$, then $d(v,j) = d(v,k-1)$, for all $j >= k-1$ and all $v$.
- If we stop computing as soon as we have a $d(*,k)$ that is identical to $d(*,k-1)$ the run time becomes
  - $O(n^3)$ when adjacency matrix is used. $(O(kn^2), k<=n)$
  - $O(ne)$ when adjacency lists are used. $(O(ke), k<=n)$

# Observations

- The computation may be done in-place.

  $d(v) = \min\{d(v), \min\{d(w) + \text{length of edge } (w,v)\}\}$

  instead of

  $d(v,k) = \min\{d(v,k-1),$
  $\min\{d(w,k-1) + \text{length of edge } (w,v)\}\}$
- Following iteration $k$, $d(v,k+1) <= d(v) <= d(v,k)$
- On termination $d(v) = d(v,n-1)$.
- Space requirement becomes $O(n)$ for $d(*)$ and $p(*)$.

# Homework

- Exercise 2 @P373